DEL PEZZO SURFACES WITH MANY SYMMETRIES

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Abstract. We classify smooth del Pezzo surfaces whose α-invariant of Tian is bigger than 1.

We assume that all varieties are projective, normal, and defined over $\mathbb{C}$.

1. Introduction

Let X be a smooth Fano variety, and let G be a finite subgroup in $\mathrm{Aut}(X)$. Put

$$\mathrm{lct}_n\big(X,G\big)=\sup\left\{\lambda\in\mathbb{Q}\ \middle|\ \begin{array}{l}\text{the log pair }\Big(X,\dfrac{\lambda}{n}D\Big)\text{ is log canonical}\\ \text{for any }G\text{-invariant divisor }D\in\big|-nK_X\big|\end{array}\right\}\in\mathbb{Q}\cup\big\{+\infty\big\}$$

for every $n\in\mathbb{N}$. Then $\mathrm{lct}_n(X)\neq+\infty\iff|-nK_X|$ contains a G-invariant divisor. Put

$$\mathrm{lct}\big(X,G\big)=\inf\Big\{\mathrm{lct}_n\big(X,G\big)\ \Big|\ n\in\mathbb{N}\Big\}\in\mathbb{R},$$

and put $\mathrm{lct}(X)=\mathrm{lct}(X,G)$ in the case when G is a trivial group.

Example 1.1 ([1, Theorem 1.7])**.** Suppose that $\dim(X)=2$. Then

$$\mathrm{lct}\big(X\big)=\begin{cases}1\text{ when }K_X^2=1\text{ and }|-K_X|\text{ has no cuspidal curves,}\\ 5/6\text{ when }K_X^2=1\text{ and }|-K_X|\text{ has a cuspidal curve,}\\ 5/6\text{ when }K_X^2=2\text{ and }|-K_X|\text{ has no tacnodal curves,}\\ 3/4\text{ when }K_X^2=2\text{ and }|-K_X|\text{ has a tacnodal curve,}\\ 3/4\text{ when }X\text{ is a cubic surface in }\mathbb{P}^3\text{ without Eckardt points,}\\ 2/3\text{ when }K_X^2=4\text{ or }X\text{ is a cubic surface in }\mathbb{P}^3\text{ with an Eckardt point,}\\ 1/2\text{ when }X\cong\mathbb{P}^1\times\mathbb{P}^1\text{ or }K_X^2\in\big\{5,6\big\},\\ 1/3\text{ in the remaining cases.}\end{cases}$$

The number $\mathrm{lct}(X,G)$ plays an important role in Kähler Geometry, since

$$\mathrm{lct}\big(X,G\big)=\alpha_G\big(X\big)$$

by [3, Theorem A.3], where $\alpha_G(X)$ is the α-invariant introduced in [10].

Theorem 1.2 ([10])**.** The variety X admits a G-invariant Kähler–Einstein metric if

$$\mathrm{lct}\big(X,G\big)>\frac{\dim\big(X\big)}{\dim\big(X\big)+1}.$$

The problem of the existence of Kähler-Einstein metrics on smooth del Pezzo surfaces is solved.

Theorem 1.3 ([11])**.** If $\dim(X)=2$, then the following conditions are equivalent:

- the surface X admits a Kähler-Einstein metric,
- the surface X is not the blow up of $\mathbb{P}^2$ in one or two points.

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Let $g_0 = g_{i\bar{j}}$ be a G-invariant Kähler metric on the variety X with a Kähler form

$$\omega_0 = \frac{\sqrt{-1}}{2\pi}\sum g_{i\bar{j}} dz_i \wedge d\overline{z}_j \in \mathrm{c}_1\big(X\big),$$

and let $\omega_1, \omega_2, \ldots, \omega_m$ be Kähler forms of some G-invariant metrics on X such that

$$\begin{cases} \mathrm{Ric}\big(\omega_m\big) = \omega_{m-1}, \\ \ldots \\ \mathrm{Ric}\big(\omega_2\big) = \omega_1, \\ \mathrm{Ric}\big(\omega_1\big) = \omega_0, \end{cases} \tag{1.4}$$

and $\omega_i \in \mathrm{c}_1(X)$ for every i. By [12], a solution to (1.4) always exist.

Theorem 1.5 ([9, Theorem 3.3])**.** Suppose that $\mathrm{lct}(X, G) > 1$. Then in $C^\infty(X)$-topology

$$\lim_{m\to+\infty} \omega_m = \omega_{KE},$$

where ω_{KE} is a Kähler form of a G-invariant Kähler–Einstein metric on the variety X.

Smooth Fano varieties that satisfy all hypotheses of Theorem 1.5 do exist.

Example 1.6. If $X \cong \mathbb{P}^1$, then $\mathrm{lct}(\mathbb{P}^1, G) > 1 \iff$ either $G \cong \mathbb{A}_4$ or $G \cong \mathbb{S}_4$ or $G \cong \mathbb{A}_5$.

Theorem 1.7 ([3, Lemma 2.30])**.** Let X_1 and X_2 be smooth Fano varieties. Then

$$\mathrm{lct}\Big(X_1 \times X_2, G_1 \times G_2\Big) = \min\Big(\mathrm{lct}\big(X_1, G_1\big), \mathrm{lct}\big(X_2, G_2\big)\Big),$$

where G_1 and G_2 are finite subgroups in $\mathrm{Aut}(X_1)$ and $\mathrm{Aut}(X_2)$ respectively.

Corollary 1.8. Let G_1 and G_2 be finite subgroups in $\mathrm{Aut}(\mathbb{P}^1)$. Then

$$\mathrm{lct}\Big(\mathbb{P}^1 \times \mathbb{P}^1, G_1 \times G_2\Big) > 1 \iff G_1 \in \{\mathbb{A}_4, \mathbb{S}_4, \mathbb{A}_5\} \ni G_2.$$

The purpose of this paper is to consider the following two problems.

Problem 1.9. Describe all smooth del Pezzo surfaces that satisfy all hypotheses of Theorem 1.5.

Problem 1.10. For a smooth del Pezzo surface X that satisfy all hypotheses of Theorem 1.5, describe all finite subgroups of the group $\mathrm{Aut}(X)$ that satisfy all hypotheses of Theorem 1.5.

There exists a partial solution to Problem 1.9 (cf. Corollary 1.8).

Example 1.11 ([8], [1], [4])**.** If $\dim(X) = 2$ and $\mathrm{Aut}(X)$ is finite, then

- $\mathrm{lct}(X, \mathrm{Aut}(X)) = 2$ if X is the Clebsch cubic surface in $\mathbb{P}^3$, which can be given by
$$x^2y + xz^2 + zt^2 + tx^2 = 0 \subset \mathbb{P}^3 \cong \mathrm{Proj}\Big(\mathbb{C}[x, y, z, t]\Big),$$
- $\mathrm{lct}(X, \mathrm{Aut}(X)) = 4$ if X is the Fermat cubic surface in $\mathbb{P}^3$,
- $\mathrm{lct}(X, \mathrm{Aut}(X)) = 2$ if X is the blow up of $\mathbb{P}^2$ at four general points.

There exists a complete solution to Problem 1.10 for $\mathbb{P}^2$ (cf. Theorem 7.5).

Example 1.12 ([8], [4])**.** Suppose that $X \cong \mathbb{P}^2$. Then the following are equivalent:

- the inequality $\mathrm{lct}(X, G) > 1$ holds,
- the inequality $\mathrm{lct}(X, G) \geqslant 4/3$ holds,
- there are no G-invariant curves in $|L|$, $|2L|$, $|3L|$, where L is a line on $\mathbb{P}^2$,
- the subgroup G is conjugate to one of the following subgroups:
  - the subgroup isomorphic to $\mathbb{PSL}(2, \mathbb{F}_7)$ that leaves invariant th quartic curve
$$x^3y + y^3z + z^3x = 0 \subset \mathbb{P}^2 \cong \mathrm{Proj}\Big(\mathbb{C}[x, y, z]\Big),$$

– the subgroup isomorphic to $\mathbb{A}_6$ that leaves invariant the sextic curve

$$10x^3y^3 + 9zx^5 + 9zy^5 + 27z^6 = 45x^2y^2z^2 + 135xyz^4 \subset \mathbb{P}^2 \cong \mathrm{Proj}\Big(\mathbb{C}[x,y,z]\Big),$$

– the Hessian subgroup of order 648 (see [13]),
– an index 3 subgroup of the Hessian subgroup.

In this paper, we prove the following result, which solves Problem 1.9.

Theorem 1.13. Suppose that $\dim(X) = 2$. Then the following are equivalent:

- there exists a finite subgroup $G \subset \mathrm{Aut}(X)$ such that $\mathrm{lct}(X, G) > 1$,
- one of the following cases hold:
  – either $X \cong \mathbb{P}^2$ or $X \cong \mathbb{P}^1 \times \mathbb{P}^1$,
  – or $\mathrm{Aut}(X)$ is finite and X is one of the following surfaces:
    * a sextic surface in $\mathbb{P}(1,1,2,3)$ such that $\mathrm{Aut}(X)$ is not Abelian
    * a quartic surface in $\mathbb{P}(1,1,1,3)$ such that
    $$\mathrm{Aut}\big(X\big) \in \Big\{\mathbb{S}_4 \times \mathbb{Z}_2, \big(\mathbb{Z}_4^2 \rtimes \mathbb{S}_3\big) \times \mathbb{Z}_2, \mathbb{PSL}\big(2, \mathbb{F}_7\big) \times \mathbb{Z}_2\Big\},$$
    * either the Clebsch cubic surface or the Fermat cubic surface in $\mathbb{P}^3$,
    * an intersection of two quadrics in $\mathbb{P}^4$ such that $\mathrm{Aut}(X) \in \{\mathbb{Z}_2^4 \rtimes \mathbb{S}_3, \mathbb{Z}_2^4 \rtimes \mathbb{D}_5\}$,
    * the blow of $\mathbb{P}^2$ at four general points.

Proof. This follows from Examples 1.11 and 1.12, Corollaries 1.8, 3.14, 4.16, 5.3, 6.5 and 7.4. □

Corollary 1.14. If $\dim(X) = 2$ and $\mathrm{Aut}(X)$ is finite, then the following are equivalent:

- the inequality $\mathrm{lct}(X, \mathrm{Aut}(X)) > 1$ holds,
- the linear system $|-K_X|$ contains no $\mathrm{Aut}(X)$-invariant curves.

The proof of Theorem 1.13 is based on auxiliary results (see Theorems 3.5, 3.6, 4.4, 4.5 and 5.4) that can be used to explicitly compute the number $\mathrm{lct}(X, G)$ in many cases.

Example 1.15. Let X be a sextic hypersurface in $\mathbb{P}(1,1,2,3)$ that is given by

$$t^2 = z^3 + xy\big(x^4 - y^4\big) \subset \mathbb{P}\big(1,1,2,3\big) \cong \mathrm{Proj}\Big(\mathbb{C}[x,y,z,t]\Big),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 2$, $\mathrm{wt}(t) = 3$. Then $\mathrm{Aut}(X) \cong \mathbb{Z}_3 \times \mathbb{Z}_2{\bullet}\mathbb{S}_4$, which implies that

$$\mathrm{lct}\Big(X, \mathrm{Aut}\big(X\big)\Big) = \mathrm{lct}_2\Big(X, \mathrm{Aut}\big(X\big)\Big) = \frac{5}{3}$$

by Theorems 1.13 and 3.6, since there is a $\mathrm{Aut}(X)$-invariant cuspidal curve in $|-2K_X|$.

We decided not to solve Problem 1.10 in this paper as the required amount of computations is too big (a priori this can be done using Theorem 1.13 and Theorem 7.5).

Example 1.16. Suppose that X is the blow of $\mathbb{P}^2$ at four general points. Then $\mathrm{Aut}(X) \cong \mathbb{S}_5$ and

$$\mathrm{lct}\big(X, G\big) > 1 \iff \mathrm{lct}\big(X, G\big) = 2 \iff |G| \in \big\{60, 120\big\},$$

since it easily follows from Example 1.1, Corollary 2.16, [1, Lemma 5.7] and [1, Lemma 5.8] that

$$\mathrm{lct}\big(X, G\big) = \begin{cases} 2 \text{ if } G \cong \mathbb{S}_5, \\ 2 \text{ if } G \cong \mathbb{A}_5, \\ 1 \text{ if } G \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4, \\ 4/5 \text{ if } G \cong \mathbb{D}_5, \\ 4/5 \text{ if } G \cong \mathbb{Z}_5, \\ 1/2 \text{ if } G \text{ is a trivial group.} \end{cases}$$

Note that the number $\mathrm{lct}(X, G)$ plays an important role in Birational Geometry (see [3], [1]), but we decided not to discuss birational applications of Theorem 1.13 in this paper.

2. Preliminaries

Let X be a smooth surface, and let D be an effective $\mathbb{Q}$-divisor on X. Put

$$D=\sum_{i=1}^{r}a_iD_i,$$

where D_i is an irreducible curve, and $a_i\in\mathbb{Q}$ such that $a_i\geqslant 0$. Suppose that $B_i\ne B_j$ for $i\ne j$.

Let $\pi\colon\bar{X}\to X$ be a birational morphism such that $\bar{X}$ is smooth as well. Put $\bar{D}=\sum_{i=1}^{r}a_i\bar{D}_i$, where $\bar{D}_i$ is a proper transform of the curve D_i on the surface $\bar{X}$. Then

$$K_{\bar{X}}+\bar{D}\sim_{\mathbb{Q}}\pi^{*}\Big(K_X+D\Big)+\sum_{i=1}^{n}c_iE_i,$$

where $c_i\in\mathbb{Q}$ and E_i is a π-exceptional curve. Suppose that $\sum_{i=1}^{r}\bar{D}_i+\sum_{i=1}^{n}E_i$ is a s.n.c. divisor.

Definition 2.1. The log pair (X,D) is KLT (respectively, log canonical) if

- the inequality $a_i<1$ holds (respectively, the inequality $a_i\leqslant 1$ holds),
- the inequality $c_j>-1$ holds (respectively, the inequality $c_j\geqslant -1$ holds),

for every $i\in\{1,\ldots,r\}$ and $j\in\{1,\ldots,n\}$.

We say that (X,D) is strictly log canonical if (X,D) is log canonical and not KLT.

Remark 2.2. The log pair (X,D) is KLT $\iff$ the log pair $(\bar{X},\bar{D}-\sum_{i=1}^{n}c_iE_i)$ is KLT.

Note that Definition 2.1 has local nature and it does not depend on the choice of π.

Remark 2.3. Let $\hat{D}$ be an effective $\mathbb{Q}$-divisor on the surface X such that $(X,\hat{D})$ is KLT and

$$\hat{D}=\sum_{i=1}^{r}\hat{a}_iD_i\sim_{\mathbb{Q}}D,$$

where $\hat{a}_i$ is a non-negative rational number. Suppose that (X,D) is not KLT. Put

$$\alpha=\min\left\{\frac{a_i}{\hat{a}_i}\ \Big|\ \hat{a}_i\ne 0\right\},$$

where α is well defined and $\alpha<1$, since (X,D) is not KLT. Put

$$D'=\sum_{i=1}^{r}\frac{a_i-\alpha\hat{a}_i}{1-\alpha}D_i\sim_{\mathbb{Q}}\hat{D}\sim_{\mathbb{Q}}D,$$

and choose $k\in\{1,\ldots,r\}$ such that $\alpha=a_k/\hat{a}_k$. Then $D_k\not\subset\mathrm{Supp}(D')$ and (X,D') is not KLT.

Let P be a point of the surface X. Recall that X is smooth by assumption. Then

$$\mathrm{mult}_P\big(D\big)\geqslant 2\Longrightarrow P\in\mathrm{LCS}\big(X,D\big)\Longrightarrow\mathrm{mult}_P\big(D\big)\geqslant 1.$$

Example 2.4. If $r=4$, $a_1=1/2$, $a_2=a_3=a_4=2/5$ and

$$3\geqslant\mathrm{mult}_P\Big(D_2\cdot D_1\Big)\geqslant 2=\mathrm{mult}_P\Big(D_3\cdot D_1\Big)\geqslant\mathrm{mult}_P\Big(D_4\cdot D_1\Big)=1,$$

then the log pair (X,D) is log canonical at the point $P\in X$.

The set of non-KLT points of the log pair (X,D) is denoted by $\mathrm{LCS}(X,D)$. Put

$$\mathcal{I}\big(X,D\big)=\pi_{*}\left(\sum_{i=1}^{n}\lceil c_i\rceil E_i-\sum_{i=1}^{r}\lfloor a_i\rfloor D_i\right),$$

and let $\mathcal{L}(X,D)$ be a subscheme that corresponds to the ideal sheaf $\mathcal{I}(X,D)$. Then

$$\mathrm{LCS}\big(X,D\big)=\mathrm{Supp}\Big(\mathcal{L}\big(X,D\big)\Big).$$

Theorem 2.5 ([7, Theorem 9.4.8])**.** Let H be a nef and big $\mathbb{Q}$-divisor on X such that

$$K_X + D + H \equiv L$$

for some Cartier divisor L on the surface X. Then $H^1(\mathcal{I}(X, D) \otimes \mathcal{O}_X(L)) = 0$.

Let $\eta\colon X \to Z$ be a surjective morphism with connected fibers.

Theorem 2.6 ([6, Theorem 7.4])**.** Let F be a fiber of the morphism η. Then the locus

$$\mathrm{LCS}\Big(X, D\Big) \cap F$$

is connected if $-(K_X + D)$ is η-nef and η-big.

Corollary 2.7. If $-(K_X + D)$ is ample, then $\mathrm{LCS}(X, D)$ is connected.

Recall that $\mathcal{I}(X, D)$ is known as the multiplier ideal sheaf (see [7, Section 9.2]).

Lemma 2.8 ([6, Theorem 7.5])**.** Suppose that the log pair (X, D) is KLT in a punctured neighborhood of the point P, but the log pair (X, D) is not KLT at the point P. Then

$$\left(\sum_{i=2}^{r} a_i D_i\right) \cdot D_1 > 1$$

in the case when $P \in D_1 \setminus \mathrm{Sing}(D_1)$.

Recall that it follows from Definition 2.1 that if the log pair (X, D) is KLT in a punctured neighborhood of the point $P \in X$, then $a_i < 1$ for every $i \in \{1, \ldots, r\}$.

Theorem 2.9 ([2, Theorem 1.28])**.** In the assumptions and notation of Lemma 2.8, suppose that

$$P \in \Big(D_1 \setminus \mathrm{Sing}\big(D_1\big)\Big) \bigcap \Big(D_2 \setminus \mathrm{Sing}\big(D_2\big)\Big)$$

and the curve D_1 intersects the curve D_2 transversally at the point $P \in X$. Then

$$\left(\sum_{i=3}^{r} a_i D_i\right) \cdot D_1 \geqslant M + Aa_1 - a_2 \text{ or } \left(\sum_{i=3}^{r} a_i D_i\right) \cdot D_2 \geqslant N + Ba_2 - a_1$$

for some non-negative rational numbers $A, B, M, N, \alpha, \beta$ that satisfy the following conditions:

- $\alpha a_1 + \beta a_2 \leqslant 1$ and $A(B-1) \geqslant 1 \geqslant \max(M, N)$,
- $\alpha(A + M - 1) \geqslant A^2(B + N - 1)\beta$ and $\alpha(1 - M) + A\beta \geqslant A$,
- either $2M + AN \leqslant 2$ or $\alpha(B + 1 - MB - N) + \beta(A + 1 - AN - M) \geqslant AB - 1$.

Corollary 2.10. In the assumptions and notation of Theorem 2.9, if $6a_1 + a_2 < 4$, then

$$\left(\sum_{i=3}^{r} a_i D_i\right) \cdot D_1 > 2a_1 - a_2 \text{ or } \left(\sum_{i=3}^{r} a_i D_i\right) \cdot D_2 > 1 + \frac{3}{2}a_2 - a_1.$$

Let $\sigma\colon \tilde{X} \to X$ be a blow up of the point P, and let F be the σ-exceptional curve. Then

$$K_{\tilde{X}} + \tilde{D} \sim_{\mathbb{Q}} \sigma^*\big(K_X + D\big) + \Big(1 - \mathrm{mult}_P\big(D\big)\Big)F$$

where $\tilde{D}$ is the proper transform of the divisor D on the surface $\tilde{X}$.

Remark 2.11. Suppose that $\mathrm{mult}_P(D) < 2$, the log pair (X, D) is KLT in a punctured neighborhood of the point P, and (X, D) is not KLT at the point P. Then there is a point $Q \in F$ such that

$$\mathrm{LCS}\Big(\tilde{X},\ \tilde{D} + \big(\mathrm{mult}_P(D) - 1\big)F\Big) \cap F = Q$$

by Theorem 2.6, which implies that $\mathrm{mult}_Q(\tilde{D}) + \mathrm{mult}_P(D) \geqslant 2$.

Suppose that X is a smooth del Pezzo surface and $D \sim_{\mathbb{Q}} -\lambda K_X$ for some $\lambda \in \mathbb{Q}$.

Lemma 2.12. Suppose that $\mathrm{LCS}(X,D)$ is a non-empty finite set. Then

$$\big|\mathrm{LCS}(X,D)\big|\leqslant h^0\Big(X,\mathcal{O}_X\big(-\lceil\lambda-1\rceil K_X\big)\Big)$$

and for every point $P\in\mathrm{LCS}(X,D)$ there exists a curve $C\in|-\lceil\lambda-1\rceil K_X|$ such that

$$\mathrm{LCS}\big(X,D\big)\setminus P\subset\mathrm{Supp}\big(C\big)\not\ni P.$$

Proof. The required assertions follow from Theorem 2.5. □

Let G be a finite subgroup in $\mathrm{Aut}(X)$ such that the following two conditions are satisfied:

- a G-invariant subgroup of the group $\mathrm{Pic}(X)$ is generated by $-K_X$,
- the divisor D is G-invariant.

Remark 2.13. If G is Abelian, then $\mathrm{lct}(X,G)\leqslant 1$.

Let ξ be the smallest integer such that $|-\xi K_X|$ contains a G-invariant curve.

Lemma 2.14. If $\xi>\lambda$, then $\mathrm{LCS}(X,D)$ is zero-dimensional.

Proof. Suppose that $\mathrm{LCS}(X,D)$ is not zero-dimensional. Then

$$D=\gamma B+D',$$

where B is a G-invariant effective Weil divisor on X, γ is a rational number such that $\gamma\geqslant 1$ and D' is a G-invariant effective $\mathbb{Q}$-divisor D on the surface X. We have that

$$B\sim -nK_X$$

for some positive integer n such that $n\geqslant\xi$. Thus, we see that

$$\lambda\Big(-K_X\Big)^2=-K_X\cdot D=\gamma\Big(-K_X\cdot B\Big)+\Big(-K_X\cdot D'\Big)\geqslant\gamma\Big(-K_X\cdot B\Big)=n\gamma\Big(-K_X\Big)^2\geqslant\xi\Big(-K_X\Big)^2,$$

which implies that $\xi\leqslant\lambda$. □

Corollary 2.15. Let k be the length of the smallest G-orbit in X. Then $\mathrm{lct}(X,G)=\xi$ if

$$h^0\Big(X,\mathcal{O}_X\Big(\big(1-\xi\big)K_X\Big)\Big)<k.$$

Corollary 2.16. If X does not contain G-fixed points, then $\mathrm{lct}(X,G)\geqslant 1$.

Most of results described in this section are valid in more general settings (see [6]).

3. Double quadric cone

Let X be a smooth sextic surface in $\mathbb{P}(1,1,2,3)$. Then X can be given by an equation

$$t^2=z^3+zf_4\big(x,y\big)+f_6\big(x,y\big)\subset\mathbb{P}\big(1,1,2,3\big)\cong\mathrm{Proj}\Big(\mathbb{C}[x,y,z,t]\Big),$$

where $\mathrm{wt}(x)=\mathrm{wt}(y)=1$, $\mathrm{wt}(z)=2$, $\mathrm{wt}(t)=3$, and $f_i(x,y)$ is a form of degree i.

Remark 3.1. It follows from the smoothness of the surface X that

- a common root of the forms $f_4(x,y)$ and $f_6(x,y)$ is not a multiple root of the form $f_6(x,y)$,
- the form $f_6(x,y)$ is not a zero form.

Let τ be the involution in $\mathrm{Aut}(X)$ such that $\tau([x:y:z:t])=[x:y:z:-t]$.

Lemma 3.2 ([5, Lemma 6.18])**.** A τ-invariant subgroup in $\mathrm{Pic}(X)$ is generated by $-K_X$.

Let G be a subgroup in $\mathrm{Aut}(X)$ such that $\tau\in G$. Recall that $\mathrm{Aut}(X)$ is finite.

Lemma 3.3. There exists a G-invariant curve in $|-2K_X|$.

Proof. Let C be the curve on X that is cut out by $z=0$. Then C is G-invariant. □

Corollary 3.4. The inequality $\mathrm{lct}(X,G)\leqslant 2$ holds.

The main purpose of this section is to prove the following two results.

Theorem 3.5. Suppose that there exists a G-invariant curve in $|-K_X|$. Then

$$\mathrm{lct}\big(X,G\big)=\mathrm{lct}_1\big(X,G\big)\in\big\{5/6,1\big\}.$$

Proof. If $\mathrm{lct}_1(X,G)=5/6$, then $\mathrm{lct}(X,G)=5/6$ by Example 1.1, since $\mathrm{lct}_1(X,G)\in\{5/6,1\}$.

Suppose that $\mathrm{lct}(X,G)<\mathrm{lct}_1(X,G)=1$. Let us derive a contradiction.

There exists a G-invariant effective $\mathbb{Q}$-divisor D on the surface X such that

$$D\sim_{\mathbb{Q}}-K_X$$

and the log pair $(X,\lambda D)$ is strictly log canonical for some rational number $\lambda<\mathrm{lct}_1(X,G)$.

By Theorem 2.6 and Lemma 2.12, the locus $\mathrm{LCS}(X,\lambda D)$ consists of a single point $P\in X$ such that P is not the base point of the pencil $|-K_X|$. Then P is G-invariant.

Let C be the unique curve in the pencil $|-K_X|$ that passes through P. Then C is G-invariant, and we may assume that $C\not\subseteq\mathrm{Supp}(D)$ (see Remark 2.3). Then

$$1>\lambda=\lambda D\cdot C\geqslant\lambda\mathrm{mult}_P\big(D\big)>1,$$

which is a contradiction. □

Theorem 3.6. Suppose that there are no G-invariant curves in $|-K_X|$. Then

$$1\leqslant\mathrm{lct}\big(X,G\big)=\mathrm{lct}_2\big(X,G\big)\leqslant 2.$$

Proof. Arguing as in the proof of Theorem 3.5, we see that $\mathrm{lct}(X,G)\geqslant 1$. Then

$$1\leqslant\mathrm{lct}\big(X,G\big)\leqslant\mathrm{lct}_2\big(X,G\big)\leqslant 2$$

by Corollary 3.4. Suppose that $\mathrm{lct}(X,G)<\mathrm{lct}_2(X,G)$. Let us derive a contradiction.

There exists a G-invariant effective $\mathbb{Q}$-divisor D on the surface X such that

$$D\sim_{\mathbb{Q}}-K_X$$

and $(X,\lambda D)$ is strictly log canonical for some rational number $\lambda<\mathrm{lct}_2(X,G)$.

By Lemmata 2.14, 2.12 and 3.2, the locus $\mathrm{LCS}(X,\lambda D)\neq\varnothing$ consists of exactly two points, which are different from the base point of the pencil $|-K_X|$.

Let P_1 and P_2 be two points in $\mathrm{LCS}(X,\lambda D)$. Then

$$\mathrm{mult}_{P_1}\big(D\big)=\mathrm{mult}_{P_2}\big(D\big)\geqslant\frac{1}{\lambda}>\frac{1}{2}.$$

Let C_1 and C_2 be the curves in $|-K_X|$ such that $P_1\in C_1$ and $P_2\in C_2$. Then

$$C_1\neq C_2$$

by Lemma 2.14. Note that C_1+C_2 is G-invariant and $C_1+C_2\sim -2K_X$.

By Remark 2.3, we may assume that C_1 and C_2 are not contained in $\mathrm{Supp}(D)$. Then

$$2=D\cdot\Big(C_1+C_2\Big)\geqslant\sum_{i=1}^{2}\mathrm{mult}_{P_i}\big(D\big)\mathrm{mult}_{P_i}\big(C_i\big)\geqslant 2\mathrm{mult}_{P_1}\big(D\big)=2\mathrm{mult}_{P_2}\big(D\big)>1,$$

which implies that $\mathrm{mult}_{P_1}(D)=\mathrm{mult}_{P_2}(D)\leqslant 1$ and $\mathrm{mult}_{P_1}(C_1)=\mathrm{mult}_{P_2}(C_2)=1$.

Let $\sigma\colon\bar{X}\to X$ be the blow-up of the surface X at the points P_1 and P_2, let E_1 and E_2 be the exceptional curves of the morphism σ such that $\sigma(E_1)=P_1$ and $\sigma(E_2)=P_2$. Then

$$K_{\bar{X}}+\lambda\bar{D}+\Big(\lambda\mathrm{mult}_{P_1}\big(D\big)-1\Big)E_1+\Big(\lambda\mathrm{mult}_{P_2}\big(D\big)-1\Big)E_2\sim_{\mathbb{Q}}\sigma^*\Big(K_X+\lambda D\Big),$$

where $\bar{D}$ is the proper transform of the divisor D on the surface $\bar{X}$.

It follows from Remark 2.11 that there are points $Q_1 \in E_1$ and $Q_2 \in E_2$ such that

$$\mathrm{LCS}\Bigg(\bar{X}, \lambda\bar{D} + \Big(\lambda\mathrm{mult}_{P_1}\big(D\big) - 1\Big)E_1 + \Big(\lambda\mathrm{mult}_{P_2}\big(D\big) - 1\Big)E_2\Bigg) = \Big\{Q_1, Q_2\Big\},$$

as $\lambda\mathrm{mult}_{P_1}(D) - 1 = \lambda\mathrm{mult}_{P_2}(D) - 1 < 1$. By Remark 2.11, we have

$$\mathrm{mult}_{P_1}\big(D\big) + \mathrm{mult}_{Q_1}\big(\bar{D}\big) = \mathrm{mult}_{P_2}\big(D\big) + \mathrm{mult}_{Q_2}\big(\bar{D}\big) \geqslant \frac{2}{\lambda} > 1. \tag{3.7}$$

Note that the action of the group G on the surface X naturally lifts to an action on $\bar{X}$.

Let $\bar{C}_1$ and $\bar{C}_2$ be the proper transforms of the curves C_1 and C_2 on the surface $\bar{X}$, respectively. Then

$$1 - \mathrm{mult}_{P_1}\big(D\big) = \bar{C}_1 \cdot \bar{D} \geqslant \mathrm{mult}_{Q_1}\big(\bar{C}_1\big)\mathrm{mult}_{Q_1}\big(\bar{D}\big),$$

which implies that $Q_1 \notin \bar{C}_1$ by (3.7). Similarly, we see that $Q_2 \notin \bar{C}_2$.

Let R be a curve that is cut out on X by $t = 0$. Then $P_1 \in R \ni P_2$, since $\tau \in G$.

Let $\bar{R}$ be the proper transform of the curve R on the surface $\bar{X}$. Then

$$Q_1 = \bar{R} \cap E_1,$$

since $\bar{R} \cap E_1$ and $\bar{C}_1 \cap E_1$ are the only τ-fixed points in E_1. Similarly, we see that $Q_2 = \bar{R} \cap E_2$.

By Remark 2.3, we may assume that $\bar{R} \not\subseteq \mathrm{Supp}(\bar{D})$, since R is smooth. Then

$$\mathrm{mult}_{Q_1}\big(\bar{D}\big) + \mathrm{mult}_{Q_2}\big(\bar{D}\big) \leqslant \bar{D} \cdot \bar{R} = 3 - \mathrm{mult}_{P_1}\big(D\big) - \mathrm{mult}_{P_2}\big(D\big),$$

which implies that $\mathrm{mult}_{Q_1}(\bar{D}) + \mathrm{mult}_{P_1}(D) = \mathrm{mult}_{Q_2}(\bar{D}) + \mathrm{mult}_{P_2}(D) \leqslant 3/2$. Thus, we have

$$\frac{3}{2} \geqslant \mathrm{mult}_{Q_1}\big(\bar{D}\big) + \mathrm{mult}_{P_1}\big(D\big) = \mathrm{mult}_{Q_2}\big(\bar{D}\big) + \mathrm{mult}_{P_2}\big(D\big) \geqslant \frac{2}{\lambda} > 1. \tag{3.8}$$

The linear system $|-2K_X|$ induces a double cover $\pi\colon X \to Q$ that is branched over $\pi(R)$, where Q is an irreducible quadric cone in $\mathbb{P}^3$. Let Π_1 and Π_2 be the planes in $\mathbb{P}^3$ such that

$$\pi\big(P_1\big) \in \Pi_1 \cap \Pi_2 \ni \pi\big(P_2\big),$$

the plane Π_1 is tangent to $\pi(R)$ at $\pi(P_1)$ and Π_2 is tangent to $\pi(R)$ at $\pi(P_2)$. Then

$$\Pi_1 \not\ni \mathrm{Sing}\big(Q\big) \notin \Pi_2,$$

since C_1 and C_2 are smooth at P_1 and P_2 respectively. Then $\Pi_1 \cap Q$ and $\Pi_2 \cap Q$ are smooth.

Let Z_1 and Z_2 be curves in $|-2K_X|$ such that $\pi(Z_1) = \Pi_1 \cap Q$ and $\pi(Z_2) = \Pi_2 \cap Q$. Then

$$Z_1 + Z_2 \in \big|-4K_X\big|$$

and the curve $Z_1 + Z_2$ is G-invariant. Note that the case $Z_1 = Z_2$ is also possible.

Suppose that $Z_1 = Z_2$. It follows from Remark 2.3 that we may assume that $Z_1 \not\subset \mathrm{Supp}(D)$, as we have $Z_1 \in |-2K_X|$. It should be mentioned (we need this for Corollary 3.12) that either

$$\left(X, \frac{5}{6}Z_1\right)$$

is strictly log canonical or the log pair (X, Z_1) is strictly log canonical. Then

$$2 = Z_1 \cdot D \geqslant \mathrm{mult}_{P_1}\big(Z_1\big)\mathrm{mult}_{P_1}\big(D\big) + \mathrm{mult}_{P_2}\big(Z_1\big)\mathrm{mult}_{P_2}\big(D\big) \geqslant 2\mathrm{mult}_{P_1}\big(D\big) + 2\mathrm{mult}_{P_2}\big(D\big) \geqslant \frac{4}{\lambda} > 2,$$

by (3.7). The obtained contradiction implies that $Z_1 \neq Z_2$.

Note that $\mathrm{mult}_{P_1}(Z_1 + Z_2) = \mathrm{mult}_{P_1}(Z_1 + Z_2) = 3$ by construction. Suppose that

$$\left(X, \frac{\lambda}{4}\Big(Z_1 + Z_2\Big)\right) \tag{3.9}$$

is KLT. By Remark 2.3, we may assume that $\mathrm{Supp}(D) \cap Z_1$ and $\mathrm{Supp}(D) \cap Z_2$ are finite subsets.

Let $\bar{Z}_1$ and $\bar{Z}_2$ be the proper transforms of the curves Z_1 and Z_2 on the surface $\bar{X}$, respectively. Then

$$0 \leqslant \bar{D} \cdot \Big(\bar{Z}_1 + \bar{Z}_2\Big) = 4 - 3\Big(\mathrm{mult}_{P_1}\big(D\big) + \mathrm{mult}_{P_2}\big(D\big)\Big) = 4 - 6\mathrm{mult}_{P_1}\big(D\big) = 4 - 6\mathrm{mult}_{P_2}\big(D\big),$$

since $\mathrm{mult}_{P_1}(Z_1 + Z_2) = \mathrm{mult}_{P_2}(Z_1 + Z_2) = 3$. Then

$$\mathrm{mult}_{P_1}\big(D\big) = \mathrm{mult}_{P_2}\big(D\big) \leqslant \frac{2}{3}. \tag{3.10}$$

Let $\rho\colon \tilde{X} \to \bar{X}$ be a blow up of the surface $\bar{X}$ at the points Q_1 and Q_2, let F_1 and F_2 be the exceptional curves of the morphism ρ such that $\rho(F_1) = Q_1$ and $\rho(F_2) = Q_2$. Then

$$K_{\tilde{X}}+\lambda\tilde{D}+\sum_{i=1}^{2}\Big(\lambda\mathrm{mult}_{P_i}\big(D\big)-1\Big)\tilde{E}_i+\sum_{i=1}^{2}\Big(\lambda\mathrm{mult}_{Q_i}\big(\bar{D}\big)+\lambda\mathrm{mult}_{P_i}\big(D\big)-2\Big)F_i \sim_{\mathbb{Q}} \big(\sigma\circ\rho\big)^*\Big(K_X+\lambda D\Big),$$

where $\tilde{D}$ and $\tilde{E}_i$ are proper transforms of the divisors D and E_i on the surface $\tilde{X}$, respectively.

It follows from Remark 2.11 that there are points $O_1 \in F_1$ and $O_2 \in F_2$ such that

$$\mathrm{LCS}\Bigg(\tilde{X}, \lambda\tilde{D}+\sum_{i=1}^{2}\Big(\lambda\mathrm{mult}_{P_i}\big(D\big)-1\Big)\tilde{E}_i+\sum_{i=1}^{2}\Big(\lambda\mathrm{mult}_{Q_i}\big(\bar{D}\big)+\lambda\mathrm{mult}_{P_i}\big(D\big)-2\Big)F_i\Bigg) = \Big\{O_1, O_2\Big\},$$

as $\lambda\mathrm{mult}_{Q_1}(\bar{D}) + \lambda\mathrm{mult}_{P_1}(D) - 2 = \lambda\mathrm{mult}_{Q_2}(\bar{D}) + \lambda\mathrm{mult}_{P_2}(D) - 2 < 1$ by (3.8).

The action of the group G on the surface $\bar{X}$ naturally lifts to an action on $\tilde{X}$ such that the curves F_1 and F_2 contain exactly two points that are fixed by τ, respectively.

Let $\tilde{R}$ be the proper transform of the curve R on the surface $\tilde{X}$. Then

- either $O_1 = \tilde{E}_1 \cap F_1$ and $O_2 = \tilde{E}_2 \cap F_2$,
- or $O_1 = \tilde{R} \cap F_1$ and $O_2 = \tilde{R} \cap F_2$.

Suppose that $O_1 = \tilde{E}_1 \cap F_1$ and $O_2 = \tilde{E}_2 \cap F_2$. It follows from Lemma 2.8 that

$$2\lambda\mathrm{mult}_{P_1}\big(D\big) - 2 = \Bigg(\lambda\tilde{D} + \Big(\lambda\mathrm{mult}_{Q_1}\big(\bar{D}\big) + \lambda\mathrm{mult}_{P_1}\big(D\big) - 2\Big)F_1\Bigg) \cdot \tilde{E}_1 > 1,$$

which implies that $\mathrm{mult}_{P_1}(D) > 3/4$, which is impossible by (3.10).

Thus, we see that $O_1 = \tilde{R} \cap F_1$ and $O_2 = \tilde{R} \cap F_2$. Then

$$\mathrm{LCS}\Bigg(\tilde{X}, \lambda\tilde{D} + \sum_{i=1}^{2}\Big(\lambda\mathrm{mult}_{Q_i}\big(\bar{D}\big) + \lambda\mathrm{mult}_{P_i}\big(D\big) - 2\Big)F_i\Bigg) = \Big\{O_1, O_2\Big\},$$

since $O_1 \notin \tilde{E}_1$ and $O_2 \notin \tilde{E}_2$. Then it follows from Remark 2.11 that

$$\mathrm{mult}_{O_1}\big(\tilde{D}\big) + \mathrm{mult}_{Q_1}\big(\bar{D}\big) + \mathrm{mult}_{P_1}\big(D\big) = \mathrm{mult}_{O_2}\big(\tilde{D}\big) + \mathrm{mult}_{Q_2}\big(\bar{D}\big) + \mathrm{mult}_{P_2}\big(D\big) \geqslant \frac{3}{\lambda}, \tag{3.11}$$

as $\lambda\mathrm{mult}_{Q_i}(\bar{D}) + \lambda\mathrm{mult}_{P_i}(D) - 2 \geqslant 0$ by (3.8). But

$$3 - \Big(\mathrm{mult}_{P_1}\big(D\big) + \mathrm{mult}_{P_2}\big(D\big) + \mathrm{mult}_{Q_1}\big(\bar{D}\big) + \mathrm{mult}_{Q_2}\big(\bar{D}\big)\Big) = \tilde{R} \cdot \tilde{D} \geqslant \mathrm{mult}_{O_1}\big(\tilde{D}\big) + \mathrm{mult}_{O_2}\big(\tilde{D}\big),$$

which contradicts (3.11), since $\lambda < \mathrm{lct}_2(X, G) \leqslant 2$.

The obtained contradiction shows that (3.9) is not KLT.

It should be pointed out that we may apply all arguments we already used for our original log pair $(X, \lambda D)$ to the log pair (3.9) with one exception: we can not use (3.10). Then

$$\frac{3}{2} \geqslant \frac{\mathrm{mult}_{Q_1}\big(\bar{Z}_1 + \bar{Z}_2\big)}{4} + \frac{\mathrm{mult}_{P_1}\big(Z_1 + Z_2\big)}{4} = \frac{\mathrm{mult}_{Q_2}\big(\bar{Z}_1 + \bar{Z}_2\big)}{4} + \frac{\mathrm{mult}_{P_2}\big(Z_1 + Z_2\big)}{4} \geqslant \frac{2}{\lambda} > 1$$

by (3.8). But $\mathrm{mult}_{P_1}(Z_1+Z_2)=\mathrm{mult}_{P_2}(Z_1+Z_2)=3$. Thus, we see that

$$3\geqslant \mathrm{mult}_{Q_1}\big(\bar{Z}_1+\bar{Z}_2\big)=\mathrm{mult}_{Q_2}\big(\bar{Z}_1+\bar{Z}_2\big)\geqslant \frac{8}{\lambda}-3>1,$$

which implies that one of the following two cases holds:

- either $\mathrm{mult}_{Q_1}(\bar{Z}_1+\bar{Z}_2)=\mathrm{mult}_{Q_2}(\bar{Z}_1+\bar{Z}_2)=2$,
- or $\mathrm{mult}_{Q_1}(\bar{Z}_1+\bar{Z}_2)=\mathrm{mult}_{Q_2}(\bar{Z}_1+\bar{Z}_2)=3$.

It follows from the construction of the curves Z_1 and Z_2 that

$$\bar{Z}_2\cap E_1=\bar{C}_1\cap E_1\neq Q_1\in \bar{R}\ni Q_2\neq \bar{C}_2\cap E_2=\bar{Z}_1\cap E_2,$$

because Z_1 is smooth at the point P_2 and Z_2 is smooth at the point P_1. Hence, we must have

$$\mathrm{mult}_{Q_1}\big(\bar{Z}_1+\bar{Z}_2\big)=\mathrm{mult}_{Q_2}\big(\bar{Z}_1+\bar{Z}_2\big)=\mathrm{mult}_{Q_1}\big(\bar{Z}_1\big)=\mathrm{mult}_{Q_2}\big(\bar{Z}_2\big)=2,$$

as $2=\mathrm{mult}_{P_1}(Z_1)\geqslant \mathrm{mult}_{Q_1}(\bar{Z}_1)$ and $2=\mathrm{mult}_{P_2}(Z_2)\geqslant \mathrm{mult}_{Q_2}(\bar{Z}_2)$.

Let $\tilde{Z}_i$ be the proper transforms of the curve Z_i on the surface $\tilde{X}$. Then

$$\varnothing\neq \mathrm{LCS}\Bigg(\tilde{X},\frac{\lambda}{4}\Big(\tilde{Z}_1+\tilde{Z}_2\Big)+\frac{3\lambda-4}{4}\Big(\tilde{E}_1+\tilde{E}_2\Big)+\frac{5\lambda-8}{4}\Big(F_1+F_2\Big)\Bigg)\subsetneq F_1\cup F_2,$$

since $3\lambda/4-1<1$ and $5\lambda/4-2<1$. On the other hand, we know that

$$\tilde{Z}_1\cap\tilde{E}_1=\varnothing=\tilde{Z}_2\cap\tilde{E}_2,$$

as we have $\mathrm{mult}_{P_1}(Z_1)=\mathrm{mult}_{Q_1}(\bar{Z}_1)$ and $\mathrm{mult}_{P_2}(Z_2)=\mathrm{mult}_{Q_2}(\bar{Z}_2)$. Then

$$\mathrm{LCS}\Bigg(\tilde{X},\frac{\lambda}{4}\Big(\tilde{Z}_1+\tilde{Z}_2\Big)+\frac{5\lambda-8}{4}\Big(F_1+F_2\Big)\Bigg)=\Big\{\tilde{R}\cap F_1,\tilde{R}\cap F_2\Big\}.$$

We can put $O_1=\tilde{R}\cap F_1$ and $O_2=\tilde{R}\cap F_i$. Since $5\lambda/4-2\geqslant 0$, we must have

$$\frac{\lambda}{4}\mathrm{mult}_{O_1}\big(\tilde{Z}_1\big)+\frac{5\lambda-8}{4}=\frac{\lambda}{4}\mathrm{mult}_{O_2}\big(\tilde{Z}_2\big)+\frac{5\lambda-8}{4}\geqslant 1,$$

which implies that $\mathrm{mult}_{O_1}(\tilde{Z}_1)\geqslant 12/\lambda-5$ and $\mathrm{mult}_{O_2}(\tilde{Z}_2)\geqslant 12/\lambda-5$. Whence

$$2=\tilde{R}\cdot\Big(\tilde{Z}_1+\tilde{Z}_2\Big)\geqslant \mathrm{mult}_{O_1}\big(\tilde{Z}_1\big)+\mathrm{mult}_{O_2}\big(\tilde{Z}_2\big)\geqslant\frac{24}{\lambda}-10>2,$$

as $\lambda<2$. The obtained contradiction implies that (3.9) is KLT. In fact, we proved that

$$\Bigg(X,\frac{1}{2}\Big(Z_1+Z_2\Big)\Bigg)$$

is log canonical (this is only important for Corollary 3.12). □

Arguing as in the proof of Theorem 3.6, we obtain the following two corollaries.

Corollary 3.12. If there are no G-invariant curves in $|-K_X|$, then $\mathrm{lct}(X,G)\in\{5/3,2\}$.

Corollary 3.13. We have $\mathrm{lct}(X,G)\in\{5/6,1,5/3,2\}$.

Using description of the group $\mathrm{Aut}(X)$ (see [5]), we obtain the following result.

Corollary 3.14. The following conditions are equivalent:

- the inequality $\mathrm{lct}(X,\mathrm{Aut}(X))>1$ holds,
- either $\mathrm{lct}(X,\mathrm{Aut}(X))=5/3$ or $\mathrm{lct}(X,\mathrm{Aut}(X))=2$,
- the pencil $|-K_X|$ does not contain G-invariant curves,
- the group $\mathrm{Aut}(X)$ is not Abelian.

Let us show how to compute $\mathrm{lct}(X,G)$ in one case.

Lemma 3.15. If $f_4(x,y)=x^2y^2$ and $f_6(x,y)=x^6+y^6+x^3y^3$, then $\mathrm{lct}(X,\mathrm{Aut}(X))=2$.

Proof. Suppose that $f_4(x,y)=x^2y^2$ and $f_6(x,y)=x^6+y^6+x^3y^3$. By [5], we have

$$\mathrm{Aut}\big(X\big)\cong\mathbb{D}_6,$$

and all $\mathrm{Aut}(X)$-invariant curves in $|-2K_X|$ can be described as follows:

- an irreducible curve that is cut out on X by $z=0$ (see the proof of Lemma 3.3),
- a reducible curve that is cut out on X by $xy=0$,
- a reducible curve that is cut out on X by $x^2+y^2=0$,
- a reducible curve that is cut out on X by $x^2-y^2=0$.

One can show that $\mathrm{Aut}(X)$-invariant curves in $|-2K_X|$ have at most ordinary double points, which implies that $\mathrm{lct}(X,\mathrm{Aut}(X))=2$ by Theorem 3.6. □

4. Double plane ramified in quartic

Let X be a smooth quartic surface in $\mathbb{P}(1,1,1,2)$. Then X can be given by an equation

$$t^2=f_4\big(x,y,z\big)\subset\mathbb{P}\big(1,1,1,2\big)\cong\mathrm{Proj}\Big(\mathbb{C}[x,y,z,t]\Big),$$

where $\mathrm{wt}(x)=\mathrm{wt}(y)=\mathrm{wt}(z)=1$, $\mathrm{wt}(t)=2$, and $f_4(x,y,z)$ is a form of degree 4.

Let τ be the involution in $\mathrm{Aut}(X)$ such that $\tau([x:y:z:t])=[x:y:z:-t]$.

Lemma 4.1 ([5, Theorem 6.17])**.** A τ-invariant subgroup in $\mathrm{Pic}(X)$ is generated by $-K_X$.

Let G be a subgroup in $\mathrm{Aut}(X)$ such that $\tau\in G$. Recall that $\mathrm{Aut}(X)$ is finite.

Lemma 4.2. There exists a G-invariant curve in $|-2K_X|$.

Proof. Let R be the curve on X that is cut out by $t=0$. Then R is G-invariant. □

Corollary 4.3. The inequality $\mathrm{lct}(X,G)\leqslant 2$ holds.

The main purpose of this section is to prove the following two results.

Theorem 4.4. Suppose that there exists a G-invariant curve in $|-K_X|$. Then

$$\mathrm{lct}\big(X,G\big)=\mathrm{lct}_1\big(X,G\big)\in\big\{3/4,5/6,1\big\}.$$

Proof. One can easily check that $\mathrm{lct}_1(X,G)\in\{3/4,5/6,1\}$. It follows from Example 1.1 that

$$\mathrm{lct}\big(X,G\big)=\mathrm{lct}_1\big(X,G\big)=\frac{3}{4}$$

if $\mathrm{lct}_1(X,G)=3/4$. Suppose that $\mathrm{lct}(X,G)<\mathrm{lct}_1(X,G)$. Let us derive a contradiction.

There exists a G-invariant effective $\mathbb{Q}$-divisor D on the surface X such that

$$D\sim_{\mathbb{Q}}-K_X$$

and the log pair $\big(X,\lambda D\big)$ is strictly log canonical for some rational number $\lambda<\mathrm{lct}_1(X,G)$.

By Theorem 2.6 and Lemma 2.12, the locus $\mathrm{LCS}(X,\lambda D)$ consists of a single point $P\in X$.

Let R be the curve on X that is cut out by $t=0$. Then $P\in R$, since $\tau\in G$.

Let L be the unique curve in $|-K_X|$ such that L is singular at the point P. Then we may assume that $\mathrm{Supp}(D)$ does not contain any component of the curve L by Remark 2.3. Then

$$2=L\cdot D\geqslant\mathrm{mult}_P\big(L\big)\cdot\mathrm{mult}_P\big(D\big)\geqslant 2\mathrm{mult}_P\big(D\big)\geqslant\frac{2}{\lambda}>1.$$

which is a contradiction. □

Theorem 4.5. Suppose that there are no G-invariant curves in $|-K_X|$. Then

$$1\leqslant\mathrm{lct}\big(X,G\big)=\min\Big(\mathrm{lct}_2\big(X,G\big),\mathrm{lct}_3\big(X,G\big)\Big)\leqslant 2.$$

Proof. Arguing as in the proof of Theorem 4.4 and using Corollary 4.3, we have

$$1 \leqslant \mathrm{lct}\big(X, G\big) \leqslant \mathrm{lct}_2\big(X, G\big) \leqslant 2.$$

Suppose that $\mathrm{lct}(X, G) < \mathrm{lct}_2(X, G)$ and $\mathrm{lct}(X, G) < \mathrm{lct}_3(X, G)$. Let us derive a contradiction.

There exists a G-invariant effective $\mathbb{Q}$-divisor D on the surface X such that

$$D \sim_{\mathbb{Q}} -K_X$$

and $(X, \lambda D)$ is strictly log canonical for some $\lambda \in \mathbb{Q}$ such that $\lambda < \mathrm{lct}_2(X, G)$ and $\lambda < \mathrm{lct}_3(X, G)$.

Let R be the curve on X that is cut out by $t = 0$. It follows from Lemmata 2.14 and 4.1 that

$$\mathrm{LCS}\big(X, \lambda D\big) \subset R,$$

and it follows from Lemma 2.12 that $|\mathrm{LCS}(X, \lambda D)| = 3$.

Let P_1, P_2, P_3 be three points in $\mathrm{LCS}(X, \lambda D)$. Then

$$\mathrm{mult}_{P_1}\big(D\big) = \mathrm{mult}_{P_2}\big(D\big) = \mathrm{mult}_{P_3}\big(D\big) \geqslant \frac{1}{\lambda} > \frac{1}{2}.$$

Let $\pi\colon X \to \mathbb{P}^2$ be a natural projection. Then π is a double cover ramified over the curve $\pi(R)$ and the points $\pi(P_1)$, $\pi(P_2)$, $\pi(P_3)$ are not contained in one line by Lemma 2.12.

Let L_1, L_2, L_3 be curves in $|-K_X|$ such that $P_2 \in L_1 \ni P_3$, $P_1 \in L_2 \ni P_3$, $P_1 \in L_3 \ni P_2$. Then

$$L_1 + L_2 + L_3 \sim -3K_X$$

and the divisor $L_1 + L_2 + L_3$ is G-invariant. We may assume that $\mathrm{Supp}(D)$ does not contain any components of the curves L_1, L_2, L_3 by Remark 2.3. Using [6, Proposition 8.21], we see that

$$\left(X, \frac{5}{8}\Big(L_1 + L_2 + L_3\Big)\right) \tag{4.6}$$

is log canonical (this is only important for Corollary 4.13). In fact, one can show that

$$\left(X, \frac{2}{3}\Big(L_1 + L_2 + L_3\Big)\right)$$

is log canonical $\iff$ (4.6) is KLT. Note that $\pi(L_1)$, $\pi(L_2)$, $\pi(L_3)$ are lines. We have

$$6 = D \cdot \Big(L_1 + L_2 + L_3\Big) \geqslant 2\sum_{i=1}^{3} \mathrm{mult}_{P_i}\big(D\big) = 6\mathrm{mult}_{P_1}\big(D\big) = 6\mathrm{mult}_{P_2}\big(D\big) = 6\mathrm{mult}_{P_3}\big(D\big),$$

which implies that $\mathrm{mult}_{P_1}(D) = \mathrm{mult}_{P_2}(D) = \mathrm{mult}_{P_3}(D) \leqslant 1$.

Let T_1, T_2, T_3 be the curves in $|-K_X|$ that are singular at P_1, P_2, P_3, respectively. Then

$$T_1 + T_2 + T_3 \sim -3K_X$$

and the divisor $T_1 + T_2 + T_3$ is G-invariant. We may assume that $\mathrm{Supp}(D)$ does not contain any components of the curves T_1, T_2, T_3 by Remark 2.3. Using [6, Proposition 8.21], we see that

$$\left(X, \frac{5}{8}\Big(T_1 + T_2 + T_3\Big)\right) \tag{4.7}$$

is log canonical (this is only important for Corollary 4.13). Moreover, one can show that

$$\left(X, \frac{2}{3}\Big(T_1 + T_2 + T_3\Big)\right)$$

is log canonical $\iff$ (4.7) is KLT $\iff$ $T_1 + T_2 + T_3 \neq L_1 + L_2 + L_3$.

Note that $\pi(T_1)$, $\pi(T_2)$, $\pi(T_3)$ are lines tangent to $\pi(R)$ at $\pi(P_1)$, $\pi(P_2)$, $\pi(P_3)$, respectively.

If $T_1+T_2+T_3=L_1+L_2+L_3$, then $\mathrm{mult}_{P_1}(D)=\mathrm{mult}_{P_2}(D)=\mathrm{mult}_{P_3}(D)\leqslant 2/3$, since

$$6=D\cdot\Big(L_1+L_2+L_3\Big)\geqslant 3\sum_{i=1}^{3}\mathrm{mult}_{P_i}\big(D\big)=9\mathrm{mult}_{P_1}\big(D\big)=9\mathrm{mult}_{P_2}\big(D\big)=9\mathrm{mult}_{P_3}\big(D\big).$$

Let Z_1, Z_2 and Z_3 be a curves in $|-2K_X|$ such that $\pi(Z_1)$, $\pi(Z_2)$, $\pi(Z_3)$ are conics where

$$\Big\{\pi\big(P_1\big),\pi\big(P_2\big),\pi\big(P_3\big)\Big\}\subset\pi\big(Z_1\big)\cap\pi\big(Z_2\big)\cap\pi\big(Z_3\big),$$

the conic $\pi(Z_1)$ is tangent to $\pi(R)$ at $\pi(P_2)$ and $\pi(P_3)$, the conic $\pi(Z_2)$ is tangent to $\pi(R)$ at the points $\pi(P_1)$ and $\pi(P_3)$, and $\pi(Z_3)$ is tangent to $\pi(R)$ at $\pi(P_1)$ and $\pi(P_2)$. Then

$$Z_1+Z_2+Z_3=2\Big(T_1+T_2+T_3\Big)\iff T_1+T_2+T_3=L_1+L_2+L_3$$

and the conics $\pi(Z_1)$, $\pi(Z_2)$, $\pi(Z_3)$ are irreducible $\iff T_1+T_2+T_3\neq L_1+L_2+L_3$. Then

$$\left(X,\frac{1}{3}\Big(Z_1+Z_2+Z_3\Big)\right)$$

is log canonical if $T_1+T_2+T_3\neq L_1+L_2+L_3$ (see Example 2.4 and [6, Proposition 8.21]). However

$$Z_1+Z_2+Z_3\sim -6K_X$$

and the divisor $Z_1+Z_2+Z_3$ is G-invariant. Thus, we may assume that $\mathrm{Supp}(D)$ does not contain any components of the curves Z_1, Z_2, Z_3 by Remark 2.3. Then

$$12=D\cdot\Big(Z_1+Z_2+Z_3\Big)\geqslant 5\sum_{i=1}^{3}\mathrm{mult}_{P_i}\big(D\big)=15\mathrm{mult}_{P_1}\big(D\big)=15\mathrm{mult}_{P_2}\big(D\big)=15\mathrm{mult}_{P_3}\big(D\big),$$

which implies that $\mathrm{mult}_{P_1}(D)=\mathrm{mult}_{P_2}(D)=\mathrm{mult}_{P_3}(D)\leqslant 4/5$. If $Z_1=Z_2=Z_3$, then

$$4=D\cdot Z_1=D\cdot Z_2=D\cdot Z_3\geqslant 2\sum_{i=1}^{3}\mathrm{mult}_{P_i}\big(D\big)=6\mathrm{mult}_{P_1}\big(D\big)=6\mathrm{mult}_{P_2}\big(D\big)=6\mathrm{mult}_{P_3}\big(D\big),$$

which implies that $\mathrm{mult}_{P_1}(D)=\mathrm{mult}_{P_2}(D)=\mathrm{mult}_{P_3}(D)\leqslant 2/3$.

Let $\sigma\colon\bar{X}\to X$ be the blow-up of the surface X at P_1, P_2 and P_3, let E_1, E_2 and E_3 be the exceptional curves of the blow up σ such that $\sigma(E_1)=P_1$, $\sigma(E_2)=P_2$ and $\sigma(E_3)=P_3$. Then

$$K_{\bar{X}}+\lambda\bar{D}+\sum_{i=1}^{3}\Big(\lambda\mathrm{mult}_{P_i}\big(D\big)-1\Big)E_i\sim_{\mathbb{Q}}\sigma^*\Big(K_X+\lambda D\Big),$$

where $\bar{D}$ is the proper transform of the divisor D on the surface $\bar{X}$.

It follows from Remark 2.11 that there are points $Q_1\in E_1$, $Q_2\in E_2$ and $Q_3\in E_3$ such that

$$\mathrm{LCS}\left(\bar{X},\lambda\bar{D}+\sum_{i=1}^{3}\Big(\lambda\mathrm{mult}_{P_i}\big(D\big)-1\Big)E_i\right)=\Big\{Q_1,Q_2,Q_3\Big\},$$

as $\lambda\mathrm{mult}_{P_1}(D)-1=\lambda\mathrm{mult}_{P_2}(D)-1=\lambda\mathrm{mult}_{P_3}(D)-1<1$. By Remark 2.11, we have

$$\mathrm{mult}_{P_1}\big(D\big)+\mathrm{mult}_{Q_1}\big(\bar{D}\big)=\mathrm{mult}_{P_2}\big(D\big)+\mathrm{mult}_{Q_2}\big(\bar{D}\big)=\mathrm{mult}_{P_3}\big(D\big)+\mathrm{mult}_{Q_3}\big(\bar{D}\big)\geqslant\frac{2}{\lambda}>1,\tag{4.8}$$

where $\mathrm{mult}_{Q_1}(\bar{D})=\mathrm{mult}_{Q_2}(\bar{D})=\mathrm{mult}_{Q_3}(\bar{D})$, since the divisor D is G-invariant.

Note that the action of the group G on the surface X naturally lifts to an action on $\bar{X}$.

Since the line $\pi(L_1)$ is not tangent to $\pi(R)$ at both $\pi(P_2)$ and $\pi(P_3)$, without loss of generality, we may assume that $\pi(L_1)$ intersects transversally $\pi(R)$ at $\pi(P_2)$. Similarly, we may assume that

- the line $\pi(L_2)$ intersects transversally the curve $\pi(R)$ at the point $\pi(P_3)$,
- the line $\pi(L_3)$ intersects transversally the curve $\pi(R)$ at the point $\pi(P_1)$.

Let $\bar{L}_1$, $\bar{L}_2$, $\bar{L}_3$ be the proper transforms of the curves L_1, L_2, L_3 on the surface $\bar{X}$, respectively. Then

$$2-\sum_{i=2}^{3}\mathrm{mult}_{P_i}\big(L_1\big)\mathrm{mult}_{P_i}\big(D\big)=\bar{L}_1\cdot\bar{D}\geqslant\sum_{i=2}^{3}\mathrm{mult}_{Q_i}\big(\bar{L}_1\big)\mathrm{mult}_{Q_i}\big(\bar{D}\big),$$

which implies that $Q_2\notin\bar{L}_1$ by (4.8). Similarly, we see that $Q_3\notin\bar{L}_2$ and $Q_1\notin\bar{L}_3$.

Let $\bar{R}$ be the proper transform of the curve R on the surface $\bar{X}$. Then

$$Q_1=\bar{R}\cap E_1$$

since the σ-exceptional curve E_1 contains exactly two points that are fixed by the involution τ, which are $\bar{R}\cap E_1$ and $\bar{L}_3\cap E_1$. Similarly, we see that $Q_2=\bar{R}\cap E_2$ and $Q_3=\bar{R}\cap E_3$.

By Remark 2.3, we may assume that $\bar{R}\not\subseteq\mathrm{Supp}(\bar{D})$, since R is smooth. Then

$$\sum_{i=1}^{3}\mathrm{mult}_{Q_i}\big(\bar{D}\big)\leqslant\bar{D}\cdot\bar{R}=4-\sum_{i=1}^{3}\mathrm{mult}_{P_i}\big(D\big),$$

where $\mathrm{mult}_{Q_1}(\bar{D})+\mathrm{mult}_{P_1}(D)=\mathrm{mult}_{Q_2}(\bar{D})+\mathrm{mult}_{P_2}(D)=\mathrm{mult}_{Q_3}(\bar{D})+\mathrm{mult}_{P_3}(D)$. Then

$$\mathrm{mult}_{Q_1}\big(\bar{D}\big)+\mathrm{mult}_{P_1}\big(D\big)=\mathrm{mult}_{Q_2}\big(\bar{D}\big)+\mathrm{mult}_{P_2}\big(D\big)=\mathrm{mult}_{Q_3}\big(\bar{D}\big)+\mathrm{mult}_{P_3}\big(D\big)\leqslant\frac{4}{3}.\tag{4.9}$$

Let $\rho\colon\tilde{X}\to\bar{X}$ be a blow up of the surface $\bar{X}$ at the points Q_1, Q_2, Q_3 and let F_1, F_2 and F_3 be the exceptional curves of the blow up ρ such that $\rho(F_1)=Q_1$, $\rho(F_2)=Q_2$ and $\rho(F_2)=Q_3$. Then

$$K_{\tilde{X}}+\lambda\tilde{D}+\sum_{i=1}^{3}\Big(\lambda\mathrm{mult}_{P_i}\big(D\big)-1\Big)\tilde{E}_i+\sum_{i=1}^{3}\Big(\lambda\mathrm{mult}_{Q_i}\big(\bar{D}\big)+\lambda\mathrm{mult}_{P_i}\big(D\big)-2\Big)F_i\sim_{\mathbb{Q}}\big(\sigma\circ\rho\big)^*\Big(K_X+\lambda D\Big),$$

where $\tilde{D}$ and $\tilde{E}_i$ are proper transforms of the divisors D and E_i on the surface $\tilde{X}$, respectively.

It follows from Remark 2.11 that there are points $O_1\in F_1$, $O_2\in F_2$ and $O_3\in F_3$ such that

$$\mathrm{LCS}\Bigg(\tilde{X},\lambda\tilde{D}+\sum_{i=1}^{3}\Big(\lambda\mathrm{mult}_{P_i}\big(D\big)-1\Big)\tilde{E}_i+\sum_{i=1}^{3}\Big(\lambda\mathrm{mult}_{Q_i}\big(\bar{D}\big)+\lambda\mathrm{mult}_{P_i}\big(D\big)-2\Big)F_i\Bigg)=\Big\{O_1,O_2,O_3\Big\},$$

since $\mathrm{mult}_{Q_1}(\bar{D})+\mathrm{mult}_{P_1}(D)=\mathrm{mult}_{Q_2}(\bar{D})+\mathrm{mult}_{P_2}(D)=\mathrm{mult}_{Q_3}(\bar{D})+\mathrm{mult}_{P_3}(D)\leqslant 4/3$.

The action of the group G on the surface $\bar{X}$ naturally lifts to an action on the surface $\tilde{X}$ such that every curve among the curves F_1, F_2 and F_3 contain exactly two τ-fixed points.

Let $\tilde{R}$ be the proper transform of the curve R on the surface $\tilde{X}$. Then

- either $O_1=\tilde{E}_1\cap F_1$, $O_2=\tilde{E}_2\cap F_2$ and $O_3=\tilde{E}_3\cap F_3$,
- or $O_1=\tilde{R}\cap F_1$, $O_2=\tilde{R}\cap F_2$ and $O_3=\tilde{R}\cap F_3$.

Suppose that $O_1=\tilde{R}\cap F_1$, $O_2=\tilde{R}\cap F_2$ and $O_3=\tilde{R}\cap F_3$. Then

$$\mathrm{LCS}\Bigg(\tilde{X},\lambda\tilde{D}+\sum_{i=1}^{3}\Big(\lambda\mathrm{mult}_{Q_i}\big(\bar{D}\big)+\lambda\mathrm{mult}_{P_i}\big(D\big)-2\Big)F_i\Bigg)=\Big\{O_1,O_2,O_3\Big\},$$

since $O_1\notin\tilde{E}_1$, $O_2\notin\tilde{E}_2$ and $O_3\notin\tilde{E}_3$. Then it follows from Remark 2.11 that

$$\mathrm{mult}_{O_i}\big(\tilde{D}\big)+\mathrm{mult}_{Q_i}\big(\bar{D}\big)+\mathrm{mult}_{P_i}\big(D\big)\geqslant\frac{3}{\lambda}>\frac{3}{2}\tag{4.10}$$

for every $i\in\{1,2,3\}$, where $\mathrm{mult}_{O_1}(\tilde{D})=\mathrm{mult}_{O_2}(\tilde{D})=\mathrm{mult}_{O_2}(\tilde{D})$. However

$$4-\sum_{i=1}^{3}\mathrm{mult}_{P_i}\big(D\big)+\sum_{i=1}^{3}\mathrm{mult}_{Q_i}\big(\bar{D}\big)=\tilde{R}\cdot\tilde{D}\geqslant\sum_{i=1}^{3}\mathrm{mult}_{O_i}\big(\tilde{D}\big),$$

which contradicts (4.10). Thus, we see that $O_1=\tilde{E}_1\cap F_1$, $O_2=\tilde{E}_2\cap F_2$ and $O_3=\tilde{E}_3\cap F_3$.

If $6(\lambda\mathrm{mult}_{P_1}(D)-1)+(\lambda\mathrm{mult}_{Q_1}(\bar{D})+\lambda\mathrm{mult}_{P_1}(D)-2)<4$, then we can apply Corollary 2.10 to

$$\left(\tilde{X},\lambda\tilde{D}+\Big(\lambda\mathrm{mult}_{P_1}\big(D\big)-1\Big)\tilde{E}_1+\Big(\lambda\mathrm{mult}_{Q_1}\big(\bar{D}\big)+\lambda\mathrm{mult}_{P_1}\big(D\big)-2\Big)F_1\right),$$

which immediately gives a contradiction, because

$$\lambda\tilde{D}\cdot F_1=\lambda\mathrm{mult}_{Q_1}\big(\bar{D}\big)\leqslant 1+\frac{3}{2}\Big(\lambda\mathrm{mult}_{Q_1}\big(\bar{D}\big)+\lambda\mathrm{mult}_{P_1}\big(D\big)-2\Big)-\Big(\lambda\mathrm{mult}_{P_1}\big(D\big)-1\Big)$$

and $\lambda\tilde{D}\cdot\tilde{E}_1=2(\lambda\mathrm{mult}_{P_1}(D)-1)-(\lambda\mathrm{mult}_{Q_1}(\bar{D})+\lambda\mathrm{mult}_{P_1}(D)-2)$. Hence

$$6\Big(\lambda\mathrm{mult}_{P_1}\big(D\big)-1\Big)+\Big(\lambda\mathrm{mult}_{Q_1}\big(\bar{D}\big)+\lambda\mathrm{mult}_{P_1}\big(D\big)-2\Big)\geqslant 4,$$

which implies that $7\mathrm{mult}_{P_1}(D)+\mathrm{mult}_{Q_1}(\bar{D})\geqslant 12/\lambda$. Similarly,

$$7\mathrm{mult}_{P_1}\big(D\big)+\mathrm{mult}_{Q_1}\big(\bar{D}\big)=7\mathrm{mult}_{P_2}\big(D\big)+\mathrm{mult}_{Q_2}\big(\bar{D}\big)=7\mathrm{mult}_{P_3}\big(D\big)+\mathrm{mult}_{Q_3}\big(\bar{D}\big)\geqslant\frac{12}{\lambda},\tag{4.11}$$

which implies that $\mathrm{mult}_{P_1}(D)=\mathrm{mult}_{P_2}(D)=\mathrm{mult}_{P_3}(D)>7/9$ by (4.9). Then

$$T_1+T_2+T_3\neq L_1+L_2+L_3,$$

since $\mathrm{mult}_{P_1}(D)=\mathrm{mult}_{P_2}(D)=\mathrm{mult}_{P_3}(D)\leqslant 2/3$ if $T_1+T_2+T_3=L_1+L_2+L_3$. We have

$$Z_1\neq Z_2\neq Z_3\neq Z_1,$$

since $\mathrm{mult}_{P_1}(D)=\mathrm{mult}_{P_2}(D)=\mathrm{mult}_{P_3}(D)\leqslant 2/3$ if $Z_1=Z_2=Z_3$.

Let $\mathcal{M}$ be linear subsystem in $|-3K_X|$ such that $M\in\mathcal{M}$ if $\pi(M)$ is a cubic curve such that

$$\Big\{\pi\big(P_1\big),\pi\big(P_2\big),\pi\big(P_3\big)\Big\}\subset\pi\big(M\big)$$

and $\pi(M)$ is tangent to $\pi(R)$ at the points $\pi(P_1)$, $\pi(P_2)$ and $\pi(P_3)$. Then

$$T_1+T_2+T_3\in\mathcal{M}\ni L_1+L_2+L_3$$

and every curve in $\mathcal{M}$ is singular at the points P_1, P_2 and P_3. Note that $\dim(\mathcal{M})\geqslant 3$.

Let $\bar{\mathcal{M}}$ be the proper transform of the linear system $\mathcal{M}$ on the surface $\bar{X}$. Then

$$\bar{\mathcal{M}}\sim\sigma^*\Big(-3K_X\Big)-\sum_{i=1}^{3}\mathrm{mult}_{P_i}\big(\mathcal{M}\big)E_3,$$

where $\mathrm{mult}_{P_1}(\mathcal{M})=\mathrm{mult}_{P_2}(\mathcal{M})=\mathrm{mult}_{P_3}(\mathcal{M})\geqslant 2$.

Let $\bar{\mathcal{B}}$ be a linear subsystem of the linear system $\bar{\mathcal{M}}$ consisting of curves that pass through the points Q_1, Q_2 and Q_3. Then $\bar{\mathcal{B}}\neq\varnothing$, since $\dim(\mathcal{M})\geqslant 3$. Put $\mathcal{B}=\sigma(\bar{\mathcal{B}})$. Then

$$\bar{\mathcal{B}}\sim\sigma^*\Big(-3K_X\Big)-\sum_{i=1}^{3}\mathrm{mult}_{P_i}\big(\mathcal{B}\big)E_3,$$

where $\mathrm{mult}_{P_1}(\mathcal{B})=\mathrm{mult}_{P_2}(\mathcal{B})=\mathrm{mult}_{P_3}(\mathcal{B})\geqslant\mathrm{mult}_{P_1}(\mathcal{M})=\mathrm{mult}_{P_2}(\mathcal{M})=\mathrm{mult}_{P_3}(\mathcal{M})\geqslant 2$.

Note that the linear systems $\mathcal{M}$, $\bar{\mathcal{B}}$ and $\mathcal{B}$ are G-invariant.

Let B be a general curve in the linear system $\mathcal{B}$. Since $|-K_X|$ contains no G-invariant curves, we see that either $\mathcal{B}=B$ or $\mathcal{B}$ has no fixed curves. If $\mathcal{B}=B$, then B is G-invariant and

$$\left(X,\frac{\lambda}{3}B\right)\tag{4.12}$$

is log canonical. Indeed, if the log pair (4.12) is not log canonical, then

$$3>\mathrm{mult}_{P_1}\big(B\big)>\frac{7}{3}>2,$$

because we can apply the arguments we used for $(X,\lambda D)$ to the log pair (4.12).

We may assume that B is not contained in $\mathrm{Supp}(D)$ by Remark 2.3. Then

$$6=B\cdot D\geqslant\sum_{i=1}^{3}\mathrm{mult}_{P_i}\big(B\big)\mathrm{mult}_{P_i}\big(D\big)>\frac{7}{9}\sum_{i=1}^{3}\mathrm{mult}_{P_i}\big(B\big)=\frac{7}{3}\mathrm{mult}_{P_1}\big(B\big)=\frac{7}{3}\mathrm{mult}_{P_2}\big(B\big)=\frac{7}{3}\mathrm{mult}_{P_1}\big(B\big),$$

which implies that $\mathrm{mult}_{P_1}(B)=\mathrm{mult}_{P_2}(B)=\mathrm{mult}_{P_3}(B)=2$.

Let $\bar{B}$ be the proper transform of the curve B on the surface $\bar{X}$. Then $\bar{B}\in\bar{\mathcal{B}}$ and

$$6-6\mathrm{mult}_{P_1}\big(D\big)=\bar{B}\cdot\bar{D}\geqslant\sum_{i=1}^{3}\mathrm{mult}_{Q_i}\big(\bar{B}\big)\mathrm{mult}_{Q_i}\big(\bar{D}\big)\geqslant 3\mathrm{mult}_{Q_1}\big(\bar{D}\big)=3\mathrm{mult}_{Q_2}\big(\bar{D}\big)=3\mathrm{mult}_{Q_3}\big(\bar{D}\big),$$

which implies that $2\mathrm{mult}_{P_1}(D)+\mathrm{mult}_{Q_1}(\bar{D})\leqslant 2$. By (4.11), we have

$$5\mathrm{mult}_{P_1}\big(D\big)+2\geqslant 7\mathrm{mult}_{P_1}\big(D\big)+\mathrm{mult}_{Q_1}\big(\bar{D}\big)\geqslant\frac{12}{\lambda}>6,$$

which implies that $\mathrm{mult}_{P_1}(D)>4/5$. But we already proved that $\mathrm{mult}_{P_1}(D)\leqslant 4/5$. □

Arguing as in the proof of Theorem 4.5, we obtain the following two corollaries.

Corollary 4.13. If there are no G-invariant curves in $|-K_X|$, then $\mathrm{lct}(X,G)\in\{15/8,2\}$.

Corollary 4.14. The equality $\mathrm{lct}(X,G)=2$ holds if the following two conditions are satisfied:

- the linear system $|-K_X|$ does not contain G-invariant curves,
- the surface X does not have G-orbits of length 3.

Corollary 4.15. We have $\mathrm{lct}(X,G)\in\{3/4,5/6,1,15/8,2\}$.

Using description of the group $\mathrm{Aut}(X)$ (see [5]), we obtain the following result.

Corollary 4.16. The following conditions are equivalent:

- the inequality $\mathrm{lct}(X,\mathrm{Aut}(X))>1$ holds,
- the equality $\mathrm{lct}(X,\mathrm{Aut}(X))=2$ holds,
- the linear system $|-K_X|$ does not contain $\mathrm{Aut}(X)$-invariant curves,
- the group $\mathrm{Aut}(X)$ is isomorphic to one of the following groups:

$$\mathbb{S}_4\times\mathbb{Z}_2,\big(\mathbb{Z}_4^2\rtimes\mathbb{S}_3\big)\times\mathbb{Z}_2,\mathbb{PSL}\big(2,\mathbb{F}_7\big)\times\mathbb{Z}_2.$$

Let us show how to compute $\mathrm{lct}(X,G)$ in two cases.

Lemma 4.17. Suppose that $f_4(x,y,z)=x^3y+y^3z+z^3x$ and $G\cong\mathbb{Z}_2\times(\mathbb{Z}_7\rtimes\mathbb{Z}_3)$. Then

$$\mathrm{lct}\big(X,G\big)=\mathrm{lct}_3\big(X,G\big)=\frac{15}{8}<\mathrm{lct}_2\big(X,G\big)=2.$$

Proof. One can easily check that the linear system $|-K_X|$ does not contain G-invariant curves, and the only G-invariant curve in $|-2K_X|$ is a curve that is cut out on X by $t=0$. Then

$$2=\mathrm{lct}_2\big(X,G\big)\geqslant\mathrm{lct}\big(X,G\big)=\min\Big(2,\mathrm{lct}_3\big(X,G\big)\Big)\in\Big\{2,15/8\Big\}$$

by Theorem 4.5 and Corollary 4.13. Note that $\mathrm{Aut}(X)\cong\mathbb{Z}_2\times\mathbb{PSL}(2,\mathbb{F}_7)$.

Put $P_1=[1:0:0:0]$, $P_2=[0:1:0:0]$, $P_3=[0:0:1:0]$. Then

- the points P_1, P_2, P_3 are contained in the unique $\mathrm{Aut}(X)$-orbit consisting of 24 points,
- the stabilizer subgroup of the subset $\{P_1,P_2,P_3\}$ is isomorphic to $\mathbb{Z}_2\times(\mathbb{Z}_7\rtimes\mathbb{Z}_3)$.

Without loss of generality, we may assume that $\{P_1,P_2,P_3\}$ is G-invariant.

The linear system $|-K_X|$ contains curves C_1, C_2 and C_3 such that

$$\mathrm{mult}_{P_1}\big(C_1\big)=\mathrm{mult}_{P_2}\big(C_2\big)=\mathrm{mult}_{P_3}\big(C_3\big)=2,$$

and the curves C_1, C_2 and C_3 have cusps at the points P_1, P_2 and P_3, respectively. Then

$$\Big(X, \frac{5}{8}\Big(C_1+C_2+C_3\Big)\Big)$$

is strictly log canonical, which implies that $\mathrm{lct}_3(X,G)\leqslant 15/8$. □

Lemma 4.18. Suppose that

$$f_4\big(x,y,z\big)=t^2+z^4+y^4+x^4+ax^2y^2+bx^2z^2+cy^2z^2,$$

where a, b and c are general complex numbers. Then $\mathrm{lct}(X,\mathrm{Aut}(X))=1$.

Proof. It follows from [5] that

$$\mathrm{Aut}\big(X\big)\cong\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2,$$

which implies that every $\mathrm{Aut}(X)$-invariant curve in $|-K_X|$ is cut out on X by one of the following equations: $x=0$, $y=0$, $z=0$. Then $\mathrm{lct}(X,\mathrm{Aut}(X))=1$ by Theorem 4.4. □

5. Cubic surfaces

Let X be a smooth cubic surface in $\mathbb{P}^3$. Then $\mathrm{Aut}(X)$ is finite. It follows from [5] that

- if $\mathrm{Aut}(X)\cong\mathbb{S}_5$, then X is the Clebsch cubic surface,
- if $\mathrm{Aut}(X)\cong\mathbb{Z}_3^2\rtimes\mathbb{S}_4$, then X is the Fermat cubic surface.

Lemma 5.1 ([1, Example 1.11])**.** If $\mathrm{Aut}(X)\cong\mathbb{S}_5$, then $\mathrm{lct}(X,\mathrm{Aut}(X))=2$.

Lemma 5.2 ([1, Lemma 5.6])**.** If $\mathrm{Aut}(X)\cong\mathbb{Z}_3^2\rtimes\mathbb{S}_4$, then $\mathrm{lct}(X,\mathrm{Aut}(X))=4$.

By [5], there is a $\mathrm{Aut}(X)$-invariant curve in $|-K_X|$ if $\mathrm{Aut}(X)\not\cong\mathbb{S}_5$ and $\mathrm{Aut}(X)\not\cong\mathbb{Z}_3^2\rtimes\mathbb{S}_4$.

Corollary 5.3. If $\mathrm{Aut}(X)\not\cong\mathbb{S}_5$ and $\mathrm{Aut}(X)\not\cong\mathbb{Z}_3^2\rtimes\mathbb{S}_4$, then $\mathrm{lct}(X,\mathrm{Aut}(X))\leqslant 1$.

The main purpose of this section is to prove the following result.

Theorem 5.4. Let G be a subgroup of the group $\mathrm{Aut}(X)$. Then

$$\mathrm{lct}\big(X,G\big)=\mathrm{lct}_1\big(X,G\big)\in\big\{2/3,5/6,1\big\}$$

if the following two conditions are satisfied:

- the linear system $|-K_X|$ contains a G-invariant curve,
- a G-invariant subgroup in $\mathrm{Pic}(X)$ is generated by $-K_X$.

Proof. Suppose $|-K_X|$ contain a G-invariant curve. Then

$$\mathrm{lct}_1\big(X,G\big)\in\big\{2/3,3/4,5/6,1\big\},$$

and it follows from Example 1.1 that $\mathrm{lct}(X,G)=\mathrm{lct}_1(X,G)=2/3$ if $\mathrm{lct}_1(X,G)=2/3$.

Suppose that a G-invariant subgroup in $\mathrm{Pic}(X)$ is $\mathbb{Z}[-K_X]$. Then $\mathrm{lct}_1(X,G)\neq 4/3$.

Suppose that $\mathrm{lct}(X,G)<\mathrm{lct}_1(X,G)\neq 2/3$. Let us derive a contradiction.

There exists a G-invariant effective $\mathbb{Q}$-divisor D on the surface X such that

$$D\sim_{\mathbb{Q}}-K_X$$

and the log pair $\big(X,\lambda D\big)$ is strictly log canonical for some rational number $\lambda<\mathrm{lct}_1(X,G)$.

By Theorem 2.6 and Lemma 2.12, the locus $\mathrm{LCS}(X,\lambda D)$ consists of a single point $P\in X$.

Let T be the curve in $|-K_X|$ such that $\mathrm{mult}_P(T)\geqslant 2$. We may assume that $\mathrm{Supp}(D)$ does not contain any component of the curve T by Remark 2.3. Then

$$3=T\cdot D\geqslant\mathrm{mult}_P\big(T\big)\cdot\mathrm{mult}_P\big(D\big)\geqslant 2\mathrm{mult}_P\big(D\big)\geqslant\frac{2}{\lambda}>1,$$

which implies $\mathrm{mult}_P(T)=2$ and $\mathrm{mult}_P(D)\leqslant 3/2$.

Note that the curve T is irreducible, which implies that $P=\mathrm{Sing}(T)$.

Let $\sigma\colon \bar{X}\to X$ be a blow up of the point P and let E be the σ-exceptional curve. Then

$$K_{\bar{X}}+\lambda\bar{D}+\Big(\lambda\mathrm{mult}_P\big(D\big)-1\Big)E\sim_{\mathbb{Q}}\sigma^*\Big(K_X+\lambda D\Big),$$

where $\bar{D}$ is the proper transform of the divisor D on the surface $\bar{X}$.

It follows from Remark 2.11 that there exists a point $Q\in E$ such that

$$\mathrm{LCS}\Bigg(\bar{X},\ \lambda\bar{D}+\Big(\lambda\mathrm{mult}_P\big(D\big)-1\Big)E\Bigg)=Q$$

and $\mathrm{mult}_Q(\bar{D})+\mathrm{mult}_P(D)\geqslant 2/\lambda$.

Let $\bar{T}$ be the proper transform of the curve T on the surface $\bar{X}$. If $Q\in\bar{T}$, then

$$3-2\mathrm{mult}_P\big(D\big)=\bar{T}\cdot\bar{D}\geqslant\mathrm{mult}_Q\big(\bar{T}\big)\mathrm{mult}_Q\big(\bar{D}\big)>\mathrm{mult}_Q\big(\bar{T}\big)\Big(2-\mathrm{mult}_P\big(D\big)\Big)\geqslant 2-\mathrm{mult}_P\big(D\big),$$

which implies that $\mathrm{mult}_P(D)\leqslant 1$. But $\mathrm{mult}_P(D)\geqslant 1/\lambda>1$. Thus, we see that $Q\not\in\bar{T}$.

As T is irreducible, the surface $\bar{X}$ is a smooth quartic hypersurface in $\mathbb{P}(1,1,1,2)$.

Let $\bar{M}$ be a general curve in $|-K_{\bar{X}}|$ such that $Q\in\bar{M}$. Then

$$3-\mathrm{mult}_P\big(D\big)=\bar{M}\cdot\bar{D}\geqslant\mathrm{mult}_Q\big(\bar{M}\big)\mathrm{mult}_Q\big(\bar{D}\big)\geqslant\mathrm{mult}_Q\big(\bar{D}\big),$$

as $\bar{M}\not\subset\mathrm{Supp}(D)$.

Let $\rho\colon\tilde{X}\to\bar{X}$ be a blow up of the point Q and let F be the ρ-exceptional curve. Then

$$K_{\tilde{X}}+\lambda\tilde{D}+\Big(\lambda\mathrm{mult}_P\big(D\big)-1\Big)\tilde{E}+\Big(\lambda\mathrm{mult}_Q\big(\bar{D}\big)+\lambda\mathrm{mult}_P\big(D\big)-2\Big)F\sim_{\mathbb{Q}}\big(\sigma\circ\rho\big)^*\Big(K_X+\lambda D\Big),$$

where $\tilde{D}$ and $\tilde{E}_i$ are proper transforms of the divisors D and E on the surface $\tilde{X}$, respectively.

It follows from Remark 2.11 that there is a point $O\in F$ such that

$$\mathrm{LCS}\Bigg(\tilde{X},\lambda\tilde{D}+\Big(\lambda\mathrm{mult}_P\big(D\big)-1\Big)\tilde{E}+\Big(\lambda\mathrm{mult}_Q\big(\bar{D}\big)+\lambda\mathrm{mult}_P\big(D\big)-2\Big)F\Bigg)=O,$$

since $\lambda\mathrm{mult}_Q(\bar{D})+\lambda\mathrm{mult}_P(D)-2\leqslant 3\lambda-2<1$. By Remark 2.11, we have

$$\mathrm{mult}_O\big(\tilde{D}\big)+\mathrm{mult}_Q\big(\bar{D}\big)+\mathrm{mult}_P\big(D\big)\geqslant\frac{3}{\lambda}>3. \tag{5.5}$$

If $O=\tilde{E}\cap F$, then it follows from Lemma 2.8 that

$$2\lambda\mathrm{mult}_P\big(D\big)-2=\Bigg(\lambda\tilde{D}+\Big(\lambda\mathrm{mult}_Q\big(\bar{D}\big)+\lambda\mathrm{mult}_P\big(D\big)-2\Big)F\Bigg)\cdot\tilde{E}>1,$$

which implies that $\mathrm{mult}_P(D)>3/2$. However $\mathrm{mult}_P(D)\leqslant 3/2$. Thus, we see that $O\not\in\tilde{E}$.

There exists a unique curve $\tilde{B}$ in the pencil $|-K_{\tilde{X}}|$ such that $O\in\tilde{B}$. Then

$$\tilde{E}\not\subset\mathrm{Supp}\big(\tilde{B}\big)\not\supset F,$$

since both $O\not\in\tilde{E}$ and $Q\not\in\bar{T}$. Put $B=\sigma\circ\rho(\tilde{B})$. Then $B\in|-K_X|$ and $B\neq T$.

The curve B is G-invariant, which implies that B is irreducible since $P\in B$.

By Remark 2.3, we may assume that $B\not\subset\mathrm{Supp}(D)$. Then

$$3-\mathrm{mult}_P\big(D\big)-\mathrm{mult}_Q\big(\bar{D}\big)=\tilde{B}\cdot\tilde{D}\geqslant\mathrm{mult}_O\big(\tilde{B}\big)\mathrm{mult}_O\big(\tilde{D}\big)\geqslant\mathrm{mult}_O\big(\tilde{D}\big),$$

which is impossible by (5.5). □

Let us show how to compute $\mathrm{lct}(X,G)$ in one case.

Lemma 5.6. Suppose that the surface X is given by the equation

$$x^3 + x\Big(y^2 + z^2 + t^2\Big) + ayzt = 0 \subset \mathbb{P}^3 \cong \mathrm{Proj}\Big(\mathbb{C}[x, y, z, t]\Big),$$

where a is a general complex number. Then $\mathrm{lct}(X, \mathrm{Aut}(X)) = 1$.

Proof. It follows from [5] that

$$\mathrm{Aut}\big(X\big) \cong \mathbb{S}_4,$$

which implies that the only $\mathrm{Aut}(X)$-invariant curve in $|-K_X|$ is cut out on X by $x = 0$.

The only $\mathrm{Aut}(X)$-invariant curve in $|-K_X|$ has ordinary double points, and $\mathrm{Aut}(X)$-invariant subgroup in $\mathrm{Pic}(X)$ is generated by $-K_X$. Then $\mathrm{lct}(X, \mathrm{Aut}(X)) = 1$ by Theorem 5.4. □

6. Intersection of two quadrics

Let X be a smooth complete intersection of two quadrics in $\mathbb{P}^4$. Then X can be given by

$$\sum_{i=0}^{4} \alpha_i x_i^2 = \sum_{i=0}^{4} \beta_i x_i^2 = 0 \subset \mathbb{P}^4 \cong \mathrm{Proj}\Big(\mathbb{C}[x_0, x_1, x_2, x_3, x_4]\Big)$$

for some $[\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 : \alpha_4] \neq [\beta_0 : \beta_1 : \beta_2 : \beta_3 : \beta_4]$ in $\mathbb{P}^4$ (see [5, Lemma 6.5]).

The group $\mathrm{Aut}(X)$ is finite. Let τ_1, τ_2, τ_3, τ_4 be involutions in $\mathrm{Aut}(X)$ such that

$$\begin{cases} \tau_1\big([x_0 : x_1 : x_2 : x_3 : x_4]\big) = [x_0 : -x_1 : x_2 : x_3 : x_4],\\ \tau_2\big([x_0 : x_1 : x_2 : x_3 : x_4]\big) = [x_0 : x_1 : -x_2 : x_3 : x_4],\\ \tau_3\big([x_0 : x_1 : x_2 : x_3 : x_4]\big) = [x_0 : x_1 : x_2 : -x_3 : x_4],\\ \tau_4\big([x_0 : x_1 : x_2 : x_3 : x_4]\big) = [x_0 : x_1 : x_2 : x_3 : -x_4], \end{cases}$$

and let Γ be a subgroup in $\mathrm{Aut}(X)$ that is generated by τ_1, τ_2, τ_3, τ_4. Then $\Gamma \cong \mathbb{Z}_2^4$.

Lemma 6.1 ([5, Theorem 6.9])**.** A Γ-invariant subgroup in $\mathrm{Pic}(X)$ is generated by $-K_X$.

The surface X contains no Γ-fixed points, which implies the following result by Corollary 2.16.

Corollary 6.2 ([1, Example 1.10])**.** The equality $\mathrm{lct}(X, \Gamma) = 1$ holds.

It easily follows from [5] that the following two conditions are equivalent:

- the linear system $|-K_X|$ does not contain $\mathrm{Aut}(X)$-invariant curves,
- either $\mathrm{Aut}(X) \cong \mathbb{Z}_2^4 \rtimes \mathbb{S}_3$ or $\mathrm{Aut}(X) \cong \mathbb{Z}_2^4 \rtimes \mathbb{D}_5$.

Corollary 6.3. If $\mathrm{Aut}(X) \not\cong \mathbb{Z}_2^4 \rtimes \mathbb{S}_3$ and $\mathrm{Aut}(X) \not\cong \mathbb{Z}_2^4 \rtimes \mathbb{D}_5$, then $\mathrm{lct}(X, \mathrm{Aut}(X)) = 1$.

The main purpose of this section is to prove the following result.

Theorem 6.4. If $\mathrm{Aut}(X) \cong \mathbb{Z}_2^4 \rtimes \mathbb{S}_3$ or $\mathrm{Aut}(X) \cong \mathbb{Z}_2^4 \rtimes \mathbb{D}_5$, then $\mathrm{lct}(X, \mathrm{Aut}(X)) = 2$.

Proof. Suppose that either $\mathrm{Aut}(X) \cong \mathbb{Z}_2^4 \rtimes \mathbb{S}_3$ or $\mathrm{Aut}(X) \cong \mathbb{Z}_2^4 \rtimes \mathbb{D}_5$. Then

$$\mathrm{lct}\big(X, G\big) \leqslant \mathrm{lct}_2\big(X, G\big) \leqslant 2,$$

since the linear system $|-2K_X|$ contains a $\mathrm{Aut}(X)$-invariant curve (see [5]).

Suppose that $\mathrm{lct}(X, G) < 2$. Let us derive a contradiction.

There exists a G-invariant effective $\mathbb{Q}$-divisor D on the surface X such that

$$D \sim_{\mathbb{Q}} -K_X$$

and $(X, \lambda D)$ is strictly log canonical for some $\lambda \in \mathbb{Q}$ such that $\lambda < 2$.

It follows from Lemmata 2.14 and 2.12 that $|\mathrm{LCS}(X, \lambda D)| \in \{2, 3, 5\}$ and $\mathrm{LCS}(X, \lambda D)$ imposes independent linear conditions on hyperplanes in $\mathbb{P}^4$, since $|-K_X|$ contains no G-invariant curves.

Suppose that $\mathrm{Aut}(X)\cong\mathbb{Z}_2^4\rtimes\mathbb{S}_3$. Then $|\mathrm{LCS}(X,\lambda D)|\neq 5$, and X can be given by

$$x_0^2+\epsilon_3x_1^2+\epsilon_3^2x_2^2+x_3^2=x_0^2+\epsilon_3^2x_1^2+\epsilon_3x_2^2+x_4^2=0\subset\mathbb{P}^4\cong\mathrm{Proj}\Big(\mathbb{C}[x_0,x_1,x_2,x_3,x_4]\Big),$$

where ϵ_3 is a primitive cube root of unity. Let ι_1 and ι_2 be elements in $\mathrm{Aut}(X)$ such that

$$\begin{cases}\iota_1\big([x_0:x_1:x_2:x_3:x_4:x_5]\big)=[x_0:x_2:x_1:x_4:x_3],\\ \iota_2([x_0:x_1:x_2:x_3:x_4:x_5])=[x_1:x_2:x_0:\epsilon_3x_3:\epsilon_3^2x_4],\end{cases}$$

and let Π be a linear subspace in $\mathbb{P}^4$ spanned by $\mathrm{LCS}(X,\lambda D)$. Then

$$\mathrm{Aut}\big(X\big)=\big\langle\Gamma,\iota_1,\iota_2\big\rangle$$

furthermore, either we have $|\mathrm{LCS}(X,\lambda D)|=2$ and Π is given by the equations $x_0=x_1=x_2=0$, or we have $|\mathrm{LCS}(X,\lambda D)|=3$ and Π is given by $x_3=x_4=0$. Since the subset

$$x_0^2+\epsilon_3x_1^2+\epsilon_3^2x_2^2+x_3^2=x_0^2+\epsilon_3^2x_1^2+\epsilon_3x_2^2+x_4^2=x_0=x_1=x_2=0$$

is empty, we have $|\mathrm{LCS}(X,\lambda D)|=3$ and Π is given by $x_3=x_4=0$. However the subset

$$x_0^2+\epsilon_3x_1^2+\epsilon_3^2x_2^2+x_3^2=x_0^2+\epsilon_3^2x_1^2+\epsilon_3x_2^2+x_4^2=x_3=x_4=0$$

consists of four points, which implies that $|\mathrm{LCS}(X,\lambda D)|\neq 3$. Thus, we have $\mathrm{Aut}(X)\not\cong\mathbb{Z}_2^4\rtimes\mathbb{S}_3$.

We see that $\mathrm{Aut}(X)\cong\mathbb{Z}_2^4\rtimes\mathbb{D}_5$. Then $|\mathrm{LCS}(X,\lambda D)|\neq 3$, and X can be given by

$$\sum_{i=0}^{4}\epsilon_5^ix_i^2=\sum_{i=0}^{4}\epsilon_5^{4-i}x_i^2=0\subset\mathbb{P}^4\cong\mathrm{Proj}\Big(\mathbb{C}[x_0,x_1,x_2,x_3,x_4]\Big)$$

where ϵ_5 is a primitive fifth root of unity. Let χ_1 and χ_2 be elements in $\mathrm{Aut}(X)$ such that

$$\begin{cases}\chi_1\big([x_0:x_1:x_2:x_3:x_4:x_5]\big)=[x_1:x_2:x_3:x_4:x_0],\\ \chi_2([x_0:x_1:x_2:x_3:x_4:x_5])=[x_4:x_3:x_2:x_1:x_0],\end{cases}$$

and let Π be a linear subspace in $\mathbb{P}^4$ spanned by $\mathrm{LCS}(X,\lambda D)$. Then

$$\mathrm{Aut}\big(X\big)=\big\langle\Gamma,\chi_1,\chi_2\big\rangle$$

and $\Pi\not\cong\mathbb{P}^1$. Since $|\mathrm{LCS}(X,\lambda D)|\in\{2,5\}$, we have $|\mathrm{LCS}(X,\lambda D)|=5$, which is impossible because the surface X does not have $\mathrm{Aut}(X)$-orbits of length 5. □

Corollary 6.5. The following four conditions are equivalent:

- the linear system $|-K_X|$ does not contain $\mathrm{Aut}(X)$-invariant curves,
- either $\mathrm{Aut}(X)\cong\mathbb{Z}_2^4\rtimes\mathbb{S}_3$ or $\mathrm{Aut}(X)\cong\mathbb{Z}_2^4\rtimes\mathbb{D}_5$,
- the inequality $\mathrm{lct}(X,\mathrm{Aut}(X))>1$ holds,
- the equality $\mathrm{lct}(X,\mathrm{Aut}(X))=2$ holds.

7. Surfaces of big degree

Let X be a smooth del Pezzo surface and let G be a finite subgroup in $\mathrm{Aut}(X)$.

Lemma 7.1. Suppose that $K_X^2=6$. Then $\mathrm{lct}(X,G)\leqslant 1$.

Proof. Let L_1, L_2, L_3, L_4, L_5 and L_6 be smooth rational curves on the surface X such that

$$L_1\cdot L_1=L_2\cdot L_2=L_3\cdot L_3=L_4\cdot L_4=L_5\cdot L_5=L_6\cdot L_6=-1$$

and $L_i\neq L_j\iff i\neq j$. Then $\sum_{i=1}^{6}L_i$ is a G-invariant curve in $|-K_X|$. □

Lemma 7.2. Suppose that $K_X^2=7$. Then $\mathrm{lct}(X,G)=1/3$.

Proof. Let L_1, L_2 and L_3 be smooth rational curves on the surface X such that

$$L_1 \cdot L_1 = L_2 \cdot L_2 = L_3 \cdot L_3 = -L_1 \cdot L_2 = -L_3 \cdot L_2 = -1$$

and $L_1 \cdot L_2 = 0$. Then $2L_1 + 3L_2 + L_1 \in |-K_X|$ and the curve $2L_1 + 3L_2 + L_1$ is G-invariant, which immediately implies that $\mathrm{lct}(X, G) = 1/3$ by Example 1.1. □

Lemma 7.3. Suppose that $K_X^2 = 8$ and $X \not\cong \mathbb{P}^1 \times \mathbb{P}^1$. Then $\mathrm{lct}(X, G) \leqslant 1/2$.

Proof. Let L and E be smooth rational curves on the surface X such that $L \cdot L = 0$ and $E \cdot E = -1$, and let C be a G-invariant curve in the linear system $|nL|$ for some $n \gg 0$. Then

$$2E + \frac{3}{n} C \sim_{\mathbb{Q}} -K_X,$$

which implies that $\mathrm{lct}(X, G) \leqslant 1/2$, since E is G-invariant. □

Corollary 7.4. If $\mathrm{lct}(X, G) > 1$ and $K_X^2 \geqslant 6$, then either $X \cong \mathbb{P}^2$ or $X \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Let us conclude this section by proving the following criterion (cf. Example 1.12).

Theorem 7.5. Suppose that $X \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then the following are equivalent:

- the inequality $\mathrm{lct}(X, G) > 1$ holds,
- the inequality $\mathrm{lct}(X, G) \geqslant 5/4$ holds,
- there are no G-invariant curves in the linear systems
$$\big|L_1\big|, \big|L_2\big|, \big|2L_1\big|, \big|2L_2\big|, \big|L_1 + L_2\big|, \big|L_1 + 2L_2\big|, \big|2L_1 + L_2\big|, \big|2L_1 + 2L_2\big|,$$
where L_1 and L_2 are fibers of two distinct natural projections of the surface X to $\mathbb{P}^1$.

Proof. Let L_1 and L_2 be fibers of two distinct natural projections of the surface X to $\mathbb{P}^1$. Then

$$\big|aL_1 + bL_2\big|$$

contains no G-invariant curves for every a and b in $\{0, 1, 2\}$ whenever $\mathrm{lct}(X, G) > 1$.

Suppose that $|L_1|$, $|L_2|$, $|2L_1|$, $|2L_2|$, $|L_1 + L_2|$, $|L_1 + 2L_2|$, $|2L_1 + L_2|$, $|2L_1 + 2L_2|$ do not contain G-invariant curves and $\mathrm{lct}(X, G) < 4/3$. Let us derive a contradiction.

There exists a G-invariant effective $\mathbb{Q}$-divisor D on the surface X such that

$$D \sim_{\mathbb{Q}} 2\big(L_1 + L_2\big) \sim -K_X$$

and $\big(X, \lambda D\big)$ is strictly log canonical for some $\lambda \in \mathbb{Q}$ such that $\lambda < 5/4$. By Theorem 2.5, we have

$$H^1\Big(X, \mathcal{I}\big(X, \lambda D\big) \otimes \mathcal{O}_X\big(L_1 + L_2\big)\Big) = 0,$$

where $\mathcal{I}(X, \lambda D)$ is the multiplier ideal sheaf of the log pair $(X, \lambda D)$ (see Section 2).

The ideal sheaf $\mathcal{I}(X, \lambda D)$ defines a zero-dimensional subscheme $\mathcal{L}$ of the surface X, since the linear system $|aL_1 + bL_2|$ has no G-invariant curves for every a and b in $\{0, 1, 2\}$.

Since the subscheme $\mathcal{L}$ is zero-dimensional, we have the short exact sequence

$$0 \longrightarrow H^0\Big(X, \mathcal{I}\big(X, \lambda D\big) \otimes \mathcal{O}_X\big(L_1 + L_2\big)\Big) \longrightarrow H^0\Big(X, \mathcal{O}_X\big(L_1 + L_2\big)\Big) \longrightarrow H^0\big(\mathcal{O}_{\mathcal{L}}\big) \longrightarrow 0,$$

which implies that $\mathrm{Supp}(\mathcal{L})$ consists of four points that are not contained in one curve in $|L_1+L_2|$.

Let P_1, P_2, P_3 and P_4 be four points in $\mathrm{Supp}(\mathcal{L})$. Then P_1, P_2, P_3 and P_4 form a G-orbit.

Write L_{11}, L_{12}, L_{13}, L_{14} for the curves in $|L_1|$ that pass through P_1, P_2, P_3, P_4, respectively, write L_{21}, L_{22}, L_{23}, L_{24} for the curves in $|L_2|$ that pass through P_1, P_2, P_3, P_4, respectively. Then

$$L_{1i} = L_{1j} \iff i = j \iff L_{2i} = L_{2j},$$

as $|L_1|$, $|L_2|$ and $|L_1 + L_2|$ do not contain G-invariant curves.

Let C_1, C_2, C_3, C_4 be the curves in the linear system $|L_1+L_2|$ such that each contains exactly three points in $\mathrm{Supp}(\mathcal{L})$ and $P_1\not\in C_1$, $P_2\not\in C_2$, $P_3\not\in C_3$, $P_4\not\in C_4$. Then

$$\Bigg(X,\frac{2}{3}\Big(C_1+C_2+C_3+C_4\Big)\Bigg)$$

is strictly log canonical, since the curves C_1, C_2, C_3, C_4 are smooth and irreducible.

By Remark 2.3, we may assume that $\mathrm{Supp}(D)$ does not contain C_1, C_2, C_3 and C_4. Then

$$16=D\cdot\Big(C_1+C_2+C_3+C_4\Big)=3\sum_{i=1}^{4}\mathrm{mult}_{P_i}\big(D\big)=12\mathrm{mult}_{P_1}\big(D\big)=\cdots=12\mathrm{mult}_{P_4}\big(D\big),$$

which implies that $\mathrm{mult}_{P_1}(D)=\mathrm{mult}_{P_2}(D)=\mathrm{mult}_{P_3}(D)=\mathrm{mult}_{P_4}(D)\leqslant 4/3$.

Let $\sigma\colon\bar{X}\to X$ be the blow-up of the points P_1, P_2, P_3 and P_4, let E_1, E_2, E_3 and E_4 be the σ-exceptional curves such that $\sigma(E_1)=P_1$, $\sigma(E_2)=P_2$, $\sigma(E_3)=P_3$ and $\sigma(E_4)=P_4$. Then

$$K_{\bar{X}}+\lambda\bar{D}+\sum_{i=1}^{4}\Big(\lambda\mathrm{mult}_{P_i}\big(D\big)-1\Big)E_i\sim_{\mathbb{Q}}\sigma^{*}\Big(K_X+\lambda D\Big),$$

where $\bar{D}$ is the proper transform of the divisor D on the surface $\bar{X}$.

By Remark 2.11, there are points $Q_1\in E_1$, $Q_2\in E_2$, $Q_3\in E_3$ and $Q_4\in E_4$ such that

$$\mathrm{LCS}\Bigg(\bar{X},\lambda\bar{D}+\sum_{i=1}^{4}\Big(\lambda\mathrm{mult}_{P_i}\big(D\big)-1\Big)E_i\Bigg)=\Big\{Q_1,Q_2,Q_3,Q_4\Big\},$$

since $\lambda\mathrm{mult}_{P_1}(D)=\mathrm{mult}_{P_2}(D)=\mathrm{mult}_{P_3}(D)=\mathrm{mult}_{P_4}(D)\leqslant 5/3<2$.

Since $\bar{D}$ is G-invariant, it follows that the action of the group G on the surface X naturally lifts to an action on $\bar{X}$ where the points Q_1, Q_2, Q_3 and Q_4 form a G-orbit.

Put $\bar{R}=3\sigma^{*}(L_1+L_2)-2\sum_{i=1}^{4}E_i$. Then $\bar{R}\cdot\bar{R}=4$, which implies that $\bar{R}$ is nef and big, since

$$\bar{L}_{11}+\bar{L}_{21}+2\bar{C}_1\sim 3\sigma^{*}\Big(L_1+L_2\Big)-2\sum_{i=1}^{4}E_i$$

and $\bar{L}_{11}\cdot\bar{R}=\bar{L}_{21}\cdot\bar{R}=1$ and $\bar{C}_1\cdot\bar{R}=0$, where we denote by symbols $\bar{L}_{11}$, $\bar{L}_{21}$ and $\bar{C}_1$ the proper transforms of the curves L_{11}, L_{21} and C_1 on the surface $\bar{X}$, respectively. Then

$$K_{\bar{X}}+\lambda\bar{D}+\sum_{i=1}^{4}\Big(\lambda\mathrm{mult}_{P_i}\big(D\big)-1\Big)E_i+\frac{1}{2}\Bigg(\bar{R}+\Big(5-4\lambda\Big)\sigma^{*}\Big(L_1+L_2\Big)\Bigg)\sim_{\mathbb{Q}}2\sigma^{*}\Big(L_1+L_2\Big)-\sum_{i=1}^{4}E_i\sim -K_{\bar{X}},$$

where $\bar{R}+(5-4\lambda)\sigma^{*}(L_1+L_2)$ is nef and big since $\lambda<5/4$. By Theorem 2.5, we have

$$H^1\Bigg(X,\mathcal{I}\Big(\bar{X},\lambda\bar{D}+\sum_{i=1}^{4}\Big(\lambda\mathrm{mult}_{P_i}\big(D\big)-1\Big)E_i\Big)\otimes\mathcal{O}_{\bar{X}}\Big(-K_{\bar{X}}\Big)\Bigg)=0,$$

from which it follows that there is a unique curve $\bar{C}\in|-K_{\bar{X}}|$ containing Q_1, Q_2, Q_3 and Q_4.

The curve $\bar{C}$ must be G-invariant, however then $\sigma(\bar{C})$ is also G-invariant, which is impossible, since $\sigma(\bar{C})\in|2L_1+2L_2|$ and $|2L_1+2L_2|$ contains no G-invariant curves. $\square$

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